QUANTIZATION OF SYMPLECTIC MANIFOLDS IN POSITIVE CHARACTERISTIC

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1. INTRODUCTION

We consider the problem of quantization: given a field k and a commutative algebra A, we wish to find an associative flat $k[[h]]$-algebra A_h such that $A_h/(h) = A$. Given such an A_h, the quotient $A_h/(h)$ carries a natural Poisson structure, via the Hayashi construction: given $a, b \in A$ with lifts $\tilde{a}, \tilde{b}$ to A_h, their Poisson bracket is defined by

$$\{a, b\} = \frac{[\tilde{a}, \tilde{b}]}{h} \mod h.$$

This is well-defined since A_h is flat over $k[[h]]$. Thus, a more refined problem of quantization is: given a *Poisson* algebra A, find an associative flat $k[[h]]$-algebra A_h such that $A_h/(h) = A$.

More generally, we may consider when A is replaced by a sheaf of functions on a space. In the smooth category, De Wilde-Lecomte and Fedosov independently proved that every symplectic manifold admits a quantization of its sheaf of smooth functions (see eg [4]). Kontsevich generalized this by showing every Poisson manifold admits a quantization of its smooth functions [8]. In this topic proposal, we consider, the problem of quantizing a symplectic manifold over a field of positive characteristic, following Bezrukavnikov and Kaledin [2]. This followed earlier work in characteristic zero [1].

Throughout, k will be a field of characteristic p. Given a vector space, scheme, ... X over k, $X^{(1)}$ will denote its base change over the Frobenius $k \to k$.

Definition 1.1. A *symplectic manifold* X/k is a smooth scheme equipped with a *symplectic form* $\Omega \in \Gamma(X, \Omega^2_X)$: Ω is closed and the map $\partial \mapsto i_\partial \Omega : \mathcal{T}_X \to \Omega^1_X$ is an isomorphism (i.e. Ω is nondegenerate).

Theorem 1.2 ([2]). *Let X/k be a symplectic manifold with symplectic form Ω. Assume that the relative Frobenius map* **Fr** *induces an isomorphism $H^i(X^{(1)}, \mathcal{O}_{X^{(1)}}) \to H^i(X, \mathcal{O}_X)$ for $i = 1, 2, 3$, and $[\Omega] \in H^2_{dR}(X)$ satisfies $C^2[\Omega] = C^1[\Omega] = 0$ as in Theorem 3.10, then there exists a quantization $\mathcal{O}_h$ of the Poisson sheaf $\mathcal{O}_X$.*

2. FROBENIUS TWIST, CARTIER ISOMORPHISM, AND p-CURVATURE

Let k be a field of characteristic p. For any scheme X/k, there is the absolute Frobenius morphism $\mathrm{Frob}_X : X \to X$ which is given on $\mathcal{O}$ by $f \mapsto f^p$. This is a lift of the absolute Frobenius on $\operatorname{Spec} k$, and thus if we define $X^{(1)}$ to be the base change of X/k over $\mathrm{Frob}_k : k \to k$, the Frobenius factors through a unique map $\mathbf{Fr}_{X/k} : X \to X^{(1)}$, the *relative Frobenius*. If the scheme X is understood, the relative Frobenius will be denoted by just **Fr**. The map

Fr is a bijection on points. If X is reduced, the map $\mathcal{O}_{X^{(1)}} \to \mathbf{Fr}_*\mathcal{O}_X$ is the embedding of the subalgebra generated by pth powers in $\mathcal{O}_X$.

Let $\Omega^\cdot_{X/S}$ denote the de Rham complex of a morphism $X \to S$; if $S = \operatorname{Spec} k$, we write $\Omega^\cdot_X = \Omega^\cdot_{X/\operatorname{Spec} k}$. The de Rham differential d is $\mathbf{Fr}_{X/k}$-linear, and thus the de Rham cohomology has the structure of an $\mathcal{O}_{X^{(1)}}$-module. The Cartier isomorphism identifies this $\mathcal{O}_{X^{(1)}}$-module. The following statement of the Cartier isomorphism is due to Katz [7].

Theorem 2.1 (Cartier)**.** *There exists a unique map* $C^{-1} : \Omega^\bullet_{X^{(1)}} \to \mathbf{Fr}_*\mathcal{H}^\bullet(\Omega_X)$ *defined by*

1. *for* f *a section of* $\mathcal{O}_X$, $C^{-1}(f \otimes 1) = f^p$*;*
2. *for* f *a section of* $\mathcal{O}_X$, $C^{-1}(d(f \otimes 1)) = f^{p-1}\,df$*;*
3. *for sections* ω, τ *of* $\Omega_{X^{(1)}}$, $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$.

Further, if X/k *is smooth, then* C^{-1} *is an isomorphism.*

Given an $\mathcal{O}_X$-module with flat connection $(\mathcal{M}, \nabla)$, we may define the p-curvature $\nabla^{(p)}$ of ∇ by

$$\nabla^{(p)}_\partial = (\nabla_\partial)^p - \nabla_{\partial^{[p]}} \in \mathcal{E}nd\,\mathcal{M}.$$

It follows from an identity of Hochschild [6, Lemma 1] that $\nabla^{(p)}_{f\partial} = f^p \nabla^{(p)}_\partial$ for all sections f of $\mathcal{O}_X$.

Lemma 2.2 ([9] Lemma 4)**.** *If* D *is a derivation of a commutative* $\mathbb{F}_p$*-algebra* A*, then for* $f \in A$*,*

$$D^{p-1}(f^{p-1}D(f)) = -D(f)^p + f^{p-1}D^p(f).$$

The following identity essentially appears in Cartier's original work on the Cartier operator [3, §II.6, Lemme 4].

Lemma 2.3 (Katz)**.** [7, Proposition 7.1.2] *If* α *is a closed* 1*-form on* X *and* $\nabla = d + \alpha$ *on* $\mathcal{O}_X$*, then for a vector field* ∂ *on* X*,*

$$\nabla^{(p)}_\partial = (\alpha(\partial))^p - (i_\partial C(\alpha))^p,$$

where we view ∂ *as a vector field on* $X^{(1)}$ *by* $\partial(f \otimes 1) = \partial(f) \otimes 1$.

Proof sketch. Use Jacobson's identity to reduce $\nabla^{(p)}_\partial$. Then the desired identity locally reduces to Lemma 2.2. □

We need one more operation throughout, which essentially is the inverse to the Cartier operation in dimension 1. Given a vector field ∂ on X/k, define $i^{[p]}_\partial : \Omega^i_X \to \Omega^{i-1}_X$ by

$$i^{[p]}_\partial(\alpha) = i_{\partial^{[p]}}(\alpha) - \mathcal{L}^{p-1}_\partial(i_\partial \alpha),$$

where $\partial^{[p]}$ is the pth power of ∂ as a derivation, and $\mathcal{L}$ is the Lie derivative.

Lemma 2.4 ([2], Lemma 2.1)**.** *For* α *a closed form,*

$$C(i^{[p]}_\partial(\alpha)) = i_\partial C(\alpha).$$

Proof sketch. First prove the identity when α is a 1-form. In that case, $i^{[p]}_\partial(\alpha) = \alpha(\partial^p) - \partial^{p-1}(\alpha(\partial))$; now the calculations of Lemma 2.3 show the desired identity. In the general case, show that $i^{[p]}_\partial$ is a derivation with respect to wedge product, up to a coboundary. □

3. Lifts of Frobenius and restricted structures

3.1. **Restricted structures.** The notion of a restricted Lie algebra is due to Jacobson.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra over k. Then $\mathfrak{g}$ is a *restricted Lie algebra* if there is a map of sets $\mathfrak{g} \to \mathfrak{g}$ denoted by $x \mapsto x^{[p]}$ such that

$$\begin{aligned} \xi : & \mathfrak{g}^{(1)} \to \mathcal{U}\mathfrak{g} \\ \xi : & x \mapsto x^p - x^{[p]} \end{aligned}$$

is a linear map into the center of $\mathcal{U}\mathfrak{g}$.

This may be written directly in terms of nonlinear identities for $x^{[p]}$. In particular, there is a universal Lie polynomial $L(x, y)$ such that

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + L(x, y).$$

3.2. **Frobenius-constant quantizations.**

Definition 3.2. Given a commutative Poisson k-algebra A, a *quantization* of A is a flat $k[[h]]$-algebra A_h complete with respect to the h-adic topology with a Poisson isomorphism $A_h/(h) \cong A$. Given a Poisson variety X/k, a *quantization* of X is a flat sheaf of $k[[h]]$-algebras $\mathcal{O}_h$ complete with respect to the h-adic topology with a Poisson isomorphism $\mathcal{O}_h/(h) \cong \mathcal{O}_X$.

Definition 3.3 ([2])**.** A quantization A_h of A is *Frobenius-constant* if there exists a central ring map

$$s : A^{(1)} \to A_h$$

such that for all $a \in A_h$,

$$s(a \mod h) - a^p \in (h^{p-1}).$$

Example 3.4. *If* $\mathfrak{g}$ *is a Lie algebra, then the (*h*-adic completion of)* $\operatorname{Rees}_h \mathcal{U}\mathfrak{g}$ *is a quantization of* $\operatorname{Sym}\mathfrak{g}$*. A set map* $-^{[p]} : \mathfrak{g} \to \mathfrak{g}$ *makes* $\mathfrak{g}$ *into a restricted Lie algebra if and only if* $s : \operatorname{Sym}\mathfrak{g}^{(1)} \to \operatorname{Rees}_h \mathcal{U}\mathfrak{g}$ *defined by*

$$s(x) = x^p - h^{p-1}x^{[p]}$$

is a central ring map. This shows $\widehat{\operatorname{Rees}_h \mathcal{U}}\mathfrak{g}$ *is a Frobenius-constant quantization of* $\mathfrak{g}$ *if* $\mathfrak{g}$ *is restricted.*

Example 3.5. *Let* $A = k[x, y]$ *be the polynomial algebra in* 2 *variables, with Poisson structure given by the symplectic form* $\Omega = dx \wedge dy$*. Then* $A_h = k[[h]]\langle x, y\rangle/([x, y] - h)$ *is a quantization of* A*, and* $s : A^{(1)} \to A_h$ *defined by* $s(x) = x^p$*,* $s(y) = y^p$ *makes* A_h *a Frobenius-constant quantization.*

3.3. **Restricted quantized algebras.** If (A_h, s) is a Frobenius-constant quantization, then Example 3.4 suggests that

$$a^{[p]} = \frac{a^p - s(a)}{h^{p-1}} \tag{1}$$

is the analog of a p-operation of a restricted Lie algebra. To produce a Frobenius-constant quantization, instead of keeping track of the splitting s through the deformation, we will keep track of the restricted structure $-^{[p]}$.

Definition 3.6. A *quantized algebra* over k is a $k[h]$ algebra A equipped with a Lie bracket $\{-,-\}$ which is a derivation in each argument and satisfies

$$h\{a,b\} = ab - ba$$

for all $a, b \in A$.

A quantized algebra where $h = 0$ is a Poisson algebra; a quantized algebra which is flat over $k[[h]]$ is a quantization.

A p-operation defined by (1) satisfies

$$(xy)^{[p]} - x^p y^{[p]} - x^{[p]} y^p + h^{p-1} x^{[p]} y^{[p]} = P(x,y), \tag{2}$$

where $P(x,y)$ is a universal quantized polynomial (i.e. a Poisson polynomial with $h\{a,b\} = ab - ba$). It is defined by

$$P(x,y) = \frac{(xy)^p - x^p y^p}{h^{p-1}}$$

in the universal quantized algebra in x and y. This motivates the following definition:

Definition 3.7. A *restricted* quantized algebra is a quantized algebra A equipped with a p-operation $a \mapsto a^{[p]}$ which makes $(A, \{-,-\})$ into a restricted Lie algebra, satisfies $h^{[p]} = h$, and satisfies (2).

If A has a Frobenius-constant quantization, then it is restricted quantized. Two of the main theorems of [2] deal with the existence and uniqueness of restricted structures. We will deal with the existence of restricted structures in Theorems 3.8 and 3.9.

Theorem 3.8 ([2] Theorem 1.11)**.** *Let X/k be a symplectic manifold with symplectic form Ω. The following are equivalent:*

1. *Hamiltonian vector fields on X are closed under pth powers;*
2. $C^2(\Omega) = 0$.

If X/k has a Frobenius-constant quantization, then both claims hold.

Proof. For f a section of $\mathcal{O}_X$, let H_f denote the Hamiltonian vector field for f. The vector field $H_f^{[p]}$ is Hamiltonian if and only if

$$C(i_{H_f^{[p]}}\Omega) = 0.$$

Now $\mathcal{L}_{H_f}^{p-1}(i_{H_f}\Omega) = \mathcal{L}_{H_f}^{p-1}(df) = 0$, so by Lemma 2.4,

$$C(i_{H_f^{[p]}}\Omega) = C(i_{H_f}^{[p]}\Omega) = i_{H_f} C(\Omega).$$

Thus, $H_f^{[p]}$ is Hamiltonian for all f if and only if $C(\Omega) = 0$. If X/k has a Frobenius-constant quantization, then $\mathcal{O}_X$ is restricted Poisson, and $H_{f^{[p]}} = H_f^{[p]}$. □

The following is an affine version of [2, Theorem 1.12], and appears there as Proposition 2.6. If as above $C^2(\Omega) = 0$, then the symplectic form Ω is locally exact. If Ω is exact, then and this allows for the construction of restricted structures.

Lemma 3.9. *Let A the coordinate ring of an affine symplectic manifold over k. Suppose the symplectic form Ω is exact and λ is a 1-form such that $d\lambda = \Omega$. Then restricted structures $-^{[p]}$ on A are in bijection with Frobenius derivations κ of A into the Poisson center of A, via the formula*

$$a^{[p]} + \kappa(a) = i_{H_a}^{[p]}(\lambda).$$

Sketch. Under Ω, λ is dual to a vector field ξ. The condition that $d\lambda = \Omega$ transports to that $\xi - \mathrm{id}$ is a derivation of the Poisson bracket. Then the various restricted quantized identities for $-^{[p]}$ follow from differentiating the universal identities for $L(x, y)$ and $P(x, y)$ with respect to ξ. □

Now recall that since the Cartier operators $C^i : \mathbf{Fr}_* \mathcal{H}^i(\Omega^\bullet_X) \to \Omega^i_X$, are defined on the cohomology sheaves, they define operations on the associated filtration on hypercohomology groups, $H_{dR}(X)$.

Theorem 3.10. *Let X/k is a symplectic manifold with symplectic form Ω, and let $[\Omega] \in H^2_{dR}(X)$ denote the image of $\Omega \in H^0(\mathcal{H}^2(\Omega^\bullet))$. If both $C^2([\Omega]) = 0$ and $C^1([\Omega]) = 0$, then X has a restricted structure compatible with Ω.*

Proof. The condition that $C^2[\Omega] = 0$ implies that Ω is locally exact. Let $\{U_\alpha\}_\alpha$ be a cover such that $\Omega|_{U_\alpha} = d\lambda_\alpha$; then $\{\lambda_\alpha - \lambda_\beta\}$ defines a Čech cohomology class in $H^1(\Omega^1_X)$. The condition that $C^1[\Omega] = 0$ is exactly that this class vanishes; that is, after a refinement, we have

$$(\lambda_\alpha - \lambda_\beta)\,|_{U_{\alpha\beta}} = d\mu_{\alpha\beta}$$

for some $\mu_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{O})$. Then $i^{[p]}_{H_a}(\lambda_\alpha - \lambda_\beta) = i^{[p]}_{H_a}(d\mu_{\alpha\beta}) = 0$. Thus, according to Lemma 3.9, we may define a restricted structure by

$$a^{[p]}|_{U_\alpha} = i^{[p]}_{H_a}(\lambda_\alpha).$$

The relation $i^{[p]}_{H_a}(\lambda_\alpha - \lambda_\beta) = 0$ shows this is well-defined. □

4. Local structure of quantization

Definition 4.1. A *quantization base* is a commutative Artinian local $k[[h]]$-algebra $(B, \mathfrak{m}_B)$ with $h \in \mathfrak{m}_B$ equipped with a derivation $K : B \to B^{(1)}$ such that $K(h) = h$.

In the above definition, $B^{(1)}$ is a B-module via the Frobenius map, and thus such a K is defined by a map $K : B \to B^{(1)}$ satisfying

$$K(ab) = a^p K(b) + K(a) b^p.$$

This is equivalent to the data of a restricted structure on B as a quantized algebra with trivial bracket, by setting $b^{[p]} = K(b)$.

Definition 4.2. A B-*quantization* of a restricted Poisson k-algebra A is a restricted quantized flat B-algebra A^B equipped with a restricted Poisson isomorphism $A^B/\mathfrak{m}_B A^B \to A$. A B-*quantization* of a restricted Poisson variety X/k is a flat B-algebra sheaf $\mathcal{O}^B$ equipped with a restricted Poisson isomorphism $\mathcal{O}^B/\mathfrak{m}_B \mathcal{O}^B \to \mathcal{O}_X$.

In order to produce a quantization as in Theorem 1.2, we will only need to consider when $B = k[h]/(h^{m+1})$ for $m \geq 0$, equipped with the Frobenius derivation K defined by $K(h) = h$. However, the proofs require reducing noncommutative deformations over such B to commutative deformations over bases B where $h = 0$.

The local model for a space X/k which we wish to quantize will be the Frobenius neighborhood. By definition, for X/k, the *Frobenius neighborhood* of a point $x \in X$ is the space

$$\mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{[p]}) \to X,$$

where $\mathfrak{m}_x^{[p]}$ is the ideal generated by pth powers of elements of $\mathfrak{m}_x$. If X/k is regular at the (locally) closed point x, then by the Cohen Structure Theorem, the completion $\hat{\mathcal{O}}_{X,x}$ is isomorphic to $k\left[\left[\mathfrak{m}_x/\mathfrak{m}_x^2\right]\right]$.

When X is symplectic, the dimension of the cotangent space at a closed point is even. For $n = \dim X/2$, the Frobenius neighborhood of a closed point is the spectrum of

$$A_F = k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p).$$

Pulling back the symplectic form of X equips $\operatorname{Spec} A_F$ with a symplectic form. When X is symplectic, so that n is even, $\operatorname{Spec} A_F$ is also equipped with a symplectic form. Our first step in understanding quantizations of $\operatorname{Spec} A_F$ is to understand the symplectic forms on $\operatorname{Spec} A_F$, up to isomorphism.

Let $\operatorname{Aut}(A_F)$ be the group scheme of automorphisms of $\operatorname{Spec} A_F$. This is an affine group scheme over k. Let $\operatorname{Aut}(A_F)_0$ denote the group subscheme of automorphisms preserving the maximal ideal $(x_1, \dots, x_n, y_1, \dots, y_n)$. The group $\operatorname{Aut}(A_F)$ is not reduced, but $\operatorname{Aut}(A_F)_0$ is.

Example 4.3. *For* $A = k[x]/x^p$, $\operatorname{Aut}(A) = \operatorname{Spec} k[a_0, a_1^{\pm 1}, a_2, \dots, a_{p-1}]/(a_0^p)$. *An* R*-point* $(a_0, \dots, a_{p-1})$ *corresponds to the automorphism of* $A \otimes_k R$ *induced by*

$$x \mapsto a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}.$$

This preserves the maximal ideal (x) *if and only if* $a_0 = 0$. *Hence*

$$\operatorname{Aut}(A)_0 = \operatorname{Spec} k[a_1^{\pm 1}, a_2, \dots, a_{p-1}].$$

To study quantizations of A_F, we study first a commutative version: we study restricted Poisson structures on $A_F \otimes B$ where B is an Artinian k-algebra. Suppose B is equipped with a Frobenius derivation $K : B \to B$ and $A_F \otimes B/B$ is equipped with a restricted Poisson structure compatible with K. First define $\operatorname{Aut}(A_F)^B = \operatorname{Res}_{B/k} \operatorname{Aut}(A_F \otimes B)$ to be the group of B-linear automorphisms of $A_F \otimes B$. Then define

$$G^B \subseteq \operatorname{Aut}(A_F)^B \qquad G_0^B \subseteq \operatorname{Aut}(A_F)_0^B$$

to be the subgroups preserving the restricted Poisson structure.

Theorem 4.4 ([2] Proposition 3.4)**.** *Let* B/k *be a quantization base. Then all nondegenerate restricted Poisson structures on* $A_F \otimes B/B$ *compatible with* K *are isomorphic.*

Vague sketch: First, the nondegenerate restricted Poisson structures form an irreducible variety. Then estimate the dimension of the $\operatorname{Aut}(A_F)^B$-orbit of a structure, using Lemma 3.9 to count the dimension of the space of restricted structures and that the Lie algebra of $\operatorname{Aut}(A_F)_0^B$ is the Lie algebra of Hamiltonian vector fields on $A_F \otimes B$. One obtains that the orbit has codimension zero.

To count dimensions via Lemma 3.9, one needs a version of the Cartier isomorphism for $A_F \otimes B/B$, which is not smooth; nonetheless there is a version. □

Theorem 4.5 ([2] Proposition 3.6)**.** *Let* $I \subseteq B \to B_0$ *be a small extension of the quantization base* B_0. *Let* A_0 *be a quantization of* A_F *over* B_0. *Then all regular quantizations* A_1 *of* A_0 *over* B *are isomorphic.*

Sketch. The idea of the above Theorem is to reduce to the case when $h = 0$, and then apply Theorem 4.4. To reduce to when $h = 0$, we consider the Baer sum $B' = B \oplus^{B_0} B$ and for $\delta : B \to B'$ the diagonal map, $B'' = B'/\delta(\mathfrak{m}_B)$. Then $B'' = B'/\delta(\mathfrak{m}_B)$ has $h = 0$, and $B' = B \oplus^k B''$. Then for A_1 and A_2 B-quantizations of A_0, $A_1 \oplus^{A_0} A_2$ is a B'-quantization of

A_0, which reduces to a B''-quantizatization, to which we may apply Theorem 4.4. Tracing through all involved identifications gives $A_1 \cong A_2$ over A_0. □

Corollary 4.6 ([2] Corollary 3.7)**.** *With notation as in Theorem 4.5, if A_1 is a B-quantization of A_0/B_0, then $\mathrm{Aut}(A_1) \to \mathrm{Aut}(A_0)$ is a surjective map of algebraic groups.*

Finally, we observe that A_F has a Frobenius-constant quantization, related to Example 3.5. Define

$$D_1 = k\,[[h]]\,\langle x, y\rangle/(x^p, y^p, xy - yx - h),$$

with splitting $s : (D_1/hD_1)^{(1)} \to D_1$ defined by $s(x) = s(y) = 0$. Then $D = D_1^{\otimes n}$ is a Frobenius-constant quantization of A_F.

5. Global story

To pass from local to global quantizations, we use the technique of formal geometry. The first step is to construct the *bundle of Frobenius frames*

$$\mathcal{M}_X = \text{“}\{\langle x, \varphi\rangle \mid x \in X, \varphi : \mathcal{O}_{X,x} \to A_F \text{ étale}\}\text{”}$$

Recalling that $\mathrm{Aut}(A_F)_0$ is the group subscheme of $\mathrm{Aut}(A_F)$ stabilizing the maximal ideal,

Proposition 5.1. $\mathcal{M}_X/\,\mathrm{Aut}(A_F)_0 = X$ *and* $\mathcal{M}_X/\,\mathrm{Aut}(A_F) = X^{(1)}$.

Now given a G-torsor $\mathcal{M}$ over Y, we have the localization functor

$$\mathrm{Loc} : G\text{−}\mathrm{rep}_{fd} \to \mathrm{Coh}(\mathcal{O}_Y).$$

This is the associated bundle construction for the principal G-bundle $\mathcal{M}$. In the context of principle bundles in flat topology, the construction takes the form of flat descent: given a flat cover $\{U_i \to Y\}$ such that $\mathcal{M} \times_Y U_i$ is trivialized,

$$\mathrm{Loc}(\mathcal{M}, V)(U_i) = \mathcal{O}_{U_i} \otimes_k V;$$

the data of a G-representation on V gives the descent datum. Localization is exact since flat descent is.

Proposition 5.2. *As a sheaf on $X^{(1)}$, $\mathrm{Loc}(\mathcal{M}_X, A_F) = \mathbf{Fr}_*\mathcal{O}_X$, where **Fr** is the relative Frobenius.*

The main idea of Bezrukavnikov and Kaledin's formal geometry is to restrict the torsor of Frobenius frames to a torsor over the structure group of a local quantization. Localizing the (local) quantization of A then gives a localization of the structure sheaf. The formal statement is as follows:

Definition 5.3 ([2] Definition 4.1)**.** Given a group G with a map $G \to \mathrm{Aut}(A_F)$, a G-*structure* is a G-torsor $\mathcal{M}_G$ on $X^{(1)}$ equipped with a map $\mathcal{M}_G \to \mathcal{M}_X$ over $G \to \mathrm{Aut}(A_F)$.

Lemma 5.4 ([2] Lemma 4.3)**.** *Let A_B be a B-quantization of A_F, with restricted quantized automorphism group G^B. If $\mathcal{M}^B$ is a G^B-structure, then $\mathrm{Loc}(\mathcal{M}^B, A_B)$ is a B-quantization of $\mathcal{O}_X$.*

Proof. Set $\mathcal{O}_B = \mathrm{Loc}(\mathcal{M}^B, A_B)$. We have identifications $\mathfrak{m}_B\mathcal{O}_B = \mathrm{Loc}(\mathcal{M}^B, \mathfrak{m}_B A_B)$ and $\mathcal{O}_B/\mathfrak{m}_B\mathcal{O}_B = \mathrm{Loc}(\mathcal{M}^B, A_B/\mathfrak{m}_B A_B)$ since the natural map pulls back to an isomorphism on a trivializing cover for $\mathcal{M}^B$. Since localization is exact, we obtain an exact sequence

$$0 \to \mathfrak{m}_B\mathcal{O}_B \to \mathcal{O}_B \to \mathcal{O}_B/\mathfrak{m}_B\mathcal{O}_B \to 0.$$

By the local criterion for flatness, $\mathcal{O}_B$ is flat over B. The action of G^B on $A_B/\mathfrak{m}_B A_B = A$ factors through Aut A, so we conclude $\mathcal{O}_B/\mathfrak{m}_B\mathcal{O}_B = \mathrm{Loc}(\mathcal{M}^B, A) \cong \mathrm{Loc}(\mathcal{M}_X, A)$. Now apply Proposition 5.2. □

The converse to the above also holds [2, Lemma 4.3].

Thus, our problem is to construct G^B-structures as B ranges over $k[h]/(h^{n+1})$ for $n \geq 0$. Given a morphism $G' \to G$ of groups and a G-torsor $\mathcal{M}_G$, a *lift* of $\mathcal{M}_G$ to G' is a G'-torsor $\mathcal{M}_{G'}$ and a map $\mathcal{M}_{G'} \to \mathcal{M}_G$ over $G' \to G$. The existence and uniqueness of lifts is governed by cohomology, according to the work of Giraud.

Theorem 5.5 ([5]). *Suppose $1 \to H \to G' \to G \to 1$ is a short exact sequence of algebraic groups over k, and H is abelian. Let $\mathcal{M}_G$ be a G-torsor on Y. Then the obstruction to lifting $\mathcal{M}_G$ to a G'-torsor is*

$$\mathrm{Loc}(\mathcal{M}_G, c) \in H^2(Y, \mathrm{Loc}(\mathcal{M}_G, H))$$

(where H is a G-representation by conjugation by G). If the obstruction vanishes, then lifts are a torsor over

$$H^1(Y, \mathrm{Loc}(\mathcal{M}_G, H)).$$

This technique is used in [2] to construct a first-order deformation. However, a first order deformation may be constructed directly.

Theorem 5.6. *If X/k is a symplectic manifold equipped with a restricted structure, then X has a $B = k[h]/h^2$-quantization given by $\mathcal{O}^B = \mathcal{O}_X \oplus h\mathcal{O}_X$ with star product*

$$f_1 \star f_2 = f_1 f_2 + \frac{h}{2}\{f_1, f_2\},$$

Poisson bracket from X, and restricted operation

$$(f + hg)^{[p]} = f^{[p]} + h(g^p + \mathrm{ad}_f^{p-1}(g)).$$

Proof. It is standard that this star product and Poisson bracket make $\mathcal{O}^B$ a B-quantization of $\mathcal{O}_X$. Observe that the restricted operation satisfies

$$(f + hg)^{[p]} = f^{[p]} + hg^p + L(f, hg),$$

where L is Jacobson's universal Lie polynomial with respect to the Poisson bracket, and that in a restricted Poisson algebra, $(ha)^{[p]} = ha^p$. As X is restricted symplectic, by Theorem 4.4, the restricted symplectic structure on the Frobenius neighborhood of a closed point $x \in X$ is isomorphic to the standard restricted symplectic structure. The Frobenius neighborhood $\mathrm{Spec}\, A_F$ with standard structure has a first-order restricted quantization given by the Moyal product on $A_F \oplus hA_F$. Hence, its p-operation satisfies $(f + hg)^{[p]} = f^{[p]} + h(g^p + \mathrm{ad}_f^{p-1}(g))$. Thus, if $x \in X$ is a closed point with ideal $\mathfrak{m}_x$, the required identities hold in $\mathcal{O}_{X,x}/\mathfrak{m}_x^{[p]}$. By Nakayama's Lemma, the required identities hold everywhere. □

Lemma 5.7 ([2] Lemma 3.10). *Let $I \subseteq B \to B_0$ be a small extension of quantization bases. The kernel of $G^B \to G^{B_0}$ is the group scheme*

$$H\langle I\rangle = \ker\left[\mathbf{Fr}_*\Omega^1_{cl}(A_F) \otimes I \xrightarrow{C - \mathbf{Fr} \otimes K} \Omega^1(A_F) \otimes I\right]$$

where $K : I \to I$ is the restriction of the restricted structure of B to I.

Sketch. The kernel of $G^B \to G^{B_0}$ is the group of transformations of the form $\mathrm{id} + D$ where $D : I \otimes_B A^B \to A^B$ is a B-derivation. The equations that D must satisfy are transferred under symplectic duality to the condition that $(C^1 - \mathbf{Fr} \otimes K)(i_D \Omega) = 0$. □

Example 5.8. *Consider* $B_0 = k[h]/(h^n)$, $B = k[h]/(h^{n+1})$, *so that* $I = (h^n)/(h^{n+1})$. *There is a unique restricted structure on* $k[h]/(h^m)$, *given by the Frobenius-derivation with* $K : h \mapsto h$. *Since* K *is a Frobenius-derivation,* $K(h^m) = mh^{(m-1)p+1}$, *so that* $K(I) = 0$ *for* $n \geq 2$. *Hence, for* $n \geq 2$ *we have*

$$H\langle I\rangle = \ker\left[C : \mathbf{Fr}_*\Omega^1_{cl}(A_F) \to \Omega^1(A_F)\right].$$

The localization of this sheaf to X *is*

$$\ker\left[C : \mathbf{Fr}_*\Omega^1_{cl}(X) \to \Omega^1(X^{(1)})\right],$$

which is exactly the sheaf of exact 1*-forms* Ω^1_{ex} *on* X.

Proof of Theorem 1.2. Let X be a symplectic manifold with symplectic form Ω, satisfying

$$\mathrm{Fr}^* : H^i(X^{(1)}, \mathcal{O}_{X^{(1)}}) \cong H^i(X, \mathcal{O}_X)$$

for $i = 1, 2, 3$, and satisfying $C^2[\Omega] = 0$ and $C^1[\Omega] = 0$. By Theorem 3.10, X admits a restricted structure. By Theorem 5.6, X has a first-order restricted quantization, which defines a $G^{k[h]/h^2}$-structure.

We have the short exact sequence

$$0 \to \mathcal{O}_{X^{(1)}} \to \mathbf{Fr}_*\mathcal{O}_X \to \mathbf{Fr}_*\Omega^1_{ex} \to 0,$$

where Ω^1_{ex} is the sheaf of exact 1-forms on X. By the long exact sequence in cohomology, $H^2(\Omega^1_{ex}) = H^1(\Omega^1_{ex}) = 0$. By Example 5.8, the obstructions to extending a $G^{k[h]/h^n}$-structure to a $G^{k[h]/h^{n+1}}$ structure is $H^2(\Omega^1_{ex})$, and such extensions are a torsor over $H^1(\Omega^1_{ex})$.

By Theorem 5.5 and Corollary 4.6, a $G^{k[h]/h^2}$-structure extends uniquely to a $G^{k[h]/h^n}$-structure for any $n \geq 2$. Now apply Lemma 5.4. □

References

[1] Roman Bezrukavnikov and Dmitry Kaledin. Fedosov quantization in algebraic context. *Moscow Mathematical Journal*, 4(3):559–592, 2004.

[2] Roman Bezrukavnikov and Dmitry Kaledin. Fedosov quantization in positive characteristic. *Journal of the American Mathematical Society*, 21(2):409–438, 2008.

[3] Pierre Cartier. Questions de rationalité des diviseurs en géométrie algébrique. *Bulletin de la Société Mathématique de France*, 86:177–251, 1958.

[4] Boris V Fedosov. A simple geometrical construction of deformation quantization. *Journal of Differential Geometry*, 40(2):213–238, 1994.

[5] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, 1971.

[6] Gerhardt Hochschild. Simple algebras with purely inseparable splitting fields of exponent 1. *Transactions of the American Mathematical Society*, 79(2):477–489, 1955.

[7] Nicholas M Katz. Algebraic solutions of differential equations (p-curvature and the Hodge filtration). *Inventiones mathematicae*, 18(1-2):1–118, 1972.

[8] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, 2003.

[9] Conjeerveram Srirangachari Seshadri. L'opération de Cartier. Applications. *Séminaire Claude Chevalley*, 4:1–26, 1958.