Math 843: Representation Theory
Lecture Notes

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Abstract. These are lecture notes for a graduate course on representation theory, taught at the University of Wisconsin-Madison in Fall 2024.

There are three main topics in these notes:

- complex representation theory of finite groups and Okounkov-Vershik's classification of representations of the symmetric group;
- Lie theory and representations of the unitary group;
- Modular representation theory and Brauer's theorem on the number of simple modular representations of a finite group.

These notes are in a preliminary form and are subject to change. If you have any comments or notice any errors, please email me.

Acknowledgments. I learned representation theory from Victor Ginzburg. His courses and style strongly influenced these notes. Madhav Nori taught me the proof of the highest weight theorem for the unitary group given here.

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CHAPTER 1

Complex representations of finite groups

1.1. (Sept 05) Introduction. Associative algebras

1.1.1. Prolegomenon. What is representation theory?

DISCIPULUS: What is representation theory?

MAGISTER: Representation theory studies algebraic objects by how they act on more linear objects. These actions are *representations.* Some basic questions of representation theory are:

- What are the irreducible representations? How many of them are there, and what do they look like?
- How is a general representation composed of irreducible representations?
- When the tensor product of representations makes sense, how does the tensor product behave?

DISCIPULUS: How may I learn representation theory?

MAGISTER: By studing the works of the masters.

DISCIPULUS: Who are the masters of representation theory, and what are their works?

Historical origins:

- Fourier: Fourier series for periodic functions (1807).
- Dirichlet's theorem on primes in arithmetic progressions (1837), Dedekind's study of the characters of the class group (1879), Frobenius's introduction of representations and characters of finite groups (1896). [1]
- Lie groups (1880s). Killing's classification of simple Lie groups (1889, Cartan 1894).
- Associative algebras? Wedderburn (1904)

Frobenius: Characters of the symmetric group (1896). Schur: representations of the general linear group (1905). Cartan: irreducible reps of ss Lie algebras (1913). Weyl: complete reducibility, classification of all representations of ss Lie algebras (1926).

Noether: reformulated Frobenius' group representations in terms of associative algebras (1929).

Brauer-Nesbitt: modular characters (1937)

Green: characters of $GL_n(\mathbf{F}_q)$. (1951)

Borel-Weil, Chevalley, Grothendieck, and others: algebro-geometric methods (algebraic groups, etc.)

Harish-Chandra: infinite-dimensional unitary representations of real reductive groups.

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[1] See [Cur99] for a more detailed discussion of Frobenius and Dedekind's exchange on this topic.

Bernstein, Harish-Chandra: p-adic reductive groups

Langlands: conjectures on representations of $G_{\mathbb{Q}}$ and number theory (L-functions, reciprocity laws...) (1967)

Deligne-Lusztig: representations of finite groups of Lie type through algebraic geometry, using étale cohomology (1976-1984).

Kazhdan-Lusztig conjectures: infinite-dimensional representation theory of complex reductive G through algebraic geometry. D-modules. Beilinson-Bernstein theorem. (1981)

Beilinson-Drinfeld: Geometric Langlands conjecture (1991).

Some modern topics (slanted towards Lie theory) are:

- infinite-dimensional representations of Lie algebras and Kac-Moody Lie algebras
- modular representations of algebraic groups
- quantum groups

DISCIPULUS: What will I learn of the masters' works in this course?

MAGISTER: Our course will focus on two concrete families of groups: the symmetric groups and the unitary groups. We will learn the finite-dimensional complex representations of both families of groups; to do so for the latter requires learning some Lie theory. We will also learn some modular representation theory, applied to the symmetric group, in order to have a taste of the non-semisimple situation.

1.1.2. Associative algebras.

See [Lor18, §1.1.1].

The term *ring* means associative unital ring: an abelian group A with an associative, bilinear product $\cdot : A \times A \to A$ and a multiplicative unit $1 \in A$. Ring homomorphisms are assumed to preserve the unit.

DEFINITION 1.1.2.1. If A is a ring, then $Z(A) = \{z \in A \mid az = za \text{ for all } a \in A\}$ is the *center* of A.

DEFINITION 1.1.2.2. Let k be a field. A *k-algebra* is a ring A and a ring homomorphism $i : k \to A$ such that $i(k) \subseteq Z(A)$.

Homomorphisms from a field are injective, so we will view $k \subseteq A$. Left multiplication by k makes A into a k-vector space.

Since $k \subseteq Z(A)$, the associative product is bilinear: if $\lambda \in k$ and $a, b \in A$, then $\lambda(ab) = (\lambda a)b = a(\lambda b)$.

EXAMPLE 1.1.2.3.

i. If X is a set, then $k\{X\}$ is the algebra of functions $X \to k$, with operations of pointwise addition and multiplication.
ii. $k\langle X\rangle$ is the free associative algebra on a set X.
iii. If V is a k-vector space, then $\mathrm{End}_k(V)$ is a k-algebra. If V is finite-dimensional, choosing a basis for V determines an isomorphism of V with the algebra $M_n(k)$ of $n \times n$ matrices with entries in k.
iv. $k[X]$ is the polynomial algebra on generators X.
v. If X is a topological space, $C(X) = \{\text{continuous } f : X \to \mathbf{C}\}$ is a $\mathbf{C}$-algebra.

If A is a k-algebra and $a \in A$, then there is an evaluation homomorphism $j_a : k[t] \to A$ which sends a polynomial p to $p(a)$.

Definition 1.1.2.4. If A is a k-algebra, then $a \in A$ is *algebraic* if there exists monic $p \in k[t]$ such that $p(a) = 0$.

If $a \in A$ is algebraic, then there exists a unique monic generator of $\ker j_a$, the *minimal polynomial* of a.

Lemma 1.1.2.5. *If $a \in A$ is algebraic with minimal polynomial p_a and $\lambda \in k$, then $p_a(\lambda) = 0$ if and only if $\lambda - a$ is not invertible.*

Proof. There exists $q(t) \in k[t]$ such that $p_a(\lambda) - p_a(t) = q(t)(\lambda - t)$. Then $p_a(\lambda) = q(a)(\lambda - a)$, so if $p_a(\lambda) = 0$, then $\lambda - a$ is a zero divisor and thus not invertible. If $p_a(\lambda) \neq 0$, then

$$1 = \frac{q(a)}{p_a(\lambda)}(\lambda - a),$$

so $\lambda - a$ is invertible. □

Definition 1.1.2.6. For $a \in A$, let $\operatorname{spec}(a) = \{\lambda \in k \mid \lambda - a \text{ not invertible in } A\}$.

We have just found that if $a \in A$ is algebraic, then $\operatorname{spec}(a)$ is the set of roots in k of the minimal polynomial of a.

A k-algebra is a k-vector space, and thus it makes sense to talk about the dimension as a k-vector space.

Definition 1.1.2.7. A k-algebra A is *nice* if

- k is algebraically closed, and
- $\dim_k A < |k|$.

Example 1.1.2.8. Finitely generated associative algebras over $\mathbf{C}$ are nice.

Example 1.1.2.9. $A = C(GL_n(\mathbf{Z}_p))$, the algebra of locally constant complex-valued functions on $GL_n(\mathbf{Z}_p)$ under convolution, is countable-dimensional over $\mathbf{C}$ and thus nice, but not finitely generated.

Theorem 1.1.2.10 (Spectral theorem. c.f. [Wal88], 0.5.1-2). *Let A be a nice k-algebra. Then if $a \in A$:*

i. a is nilpotent if and only if $\operatorname{spec}(a) = \{0\}$;
ii. a is algebraic if and only if $\operatorname{spec}(a)$ is finite and nonempty;
iii. a is not algebraic if and only if $|k| = |\operatorname{spec}(a)|$.

In particular, $\operatorname{spec}(a) \neq \varnothing$.

Proof. 2. and 3. We have already seen in Lemma 1.1.2.5 that if a is algebraic, then $\operatorname{spec}(a)$ is the set of roots in k of the minimal polynomial of a. Since k is algebraically closed, the set of roots of the minimal polynomial is nonempty.

Suppose $a \in A$ is not algebraic. Since A is nice, $k = \mathbf{C}$ and $\dim_{\mathbf{C}} A$ is at most countable. Let

$$S_a = \operatorname{span}\{\frac{1}{\lambda - t} \mid \lambda \notin \operatorname{spec}(a)\} \subseteq k(t).$$

We have an evaluation map $J_a : S_a \to A$ which sends $(\lambda - t)^{-1}$ to $(\lambda - a)^{-1}$, which is well-defined since $\lambda - a$ is invertible when $\lambda \notin \operatorname{spec}(a)$.

I claim J_a is injective: if $\sum_i \frac{c_i}{\lambda_i - a} = 0$ in A, then a satisfies the polynomial equation

$$0 = \sum_i c_i \prod_{j \neq i} (\lambda_j - a),$$

contradicting that a is not algebraic. Hence,

$$|k \setminus \operatorname{spec}(a)| = \dim_k S_a \leq \dim_k A < |k|,$$

and since k is infinite, $|\operatorname{spec}(a)| = |k|$.

Finally, a is nilpotent if and only if a is algebraic and its minimal polynomial is t^n for some $n \geq 1$. As k is algebraically closed, the set of roots of the minimal polynomial is $\{0\}$ if and only if the minimal polynomial is of the form t^n. □

Definition 1.1.2.11. A ring A is a *division ring* if all nonzero elements are units.

Lemma 1.1.2.12 (Dixmier). *If A is a nice division k-algebra, then $A = k$.*

Proof. Suppose $a \in A \setminus k$. Then $a - \lambda \notin k$ and so $a - \lambda \neq 0$ for all $\lambda \in k$, so $a - \lambda$ is invertible. Thus $\operatorname{spec}(a) = \varnothing$, contradicting Theorem 1.1.2.10. Thus $A = k$. □

1.1.3. Modules. If M is an abelian group, then $\operatorname{End}(M)$ is a ring under addition and composition. A ring homomorphism $A \to \operatorname{End}(M)$ is equivalent to a bilinear function $\cdot : A \times M \to M$ such that $(ab)m = a(bm)$ for $a, b \in A$ and $m \in M$.

Definition 1.1.3.1. An abelian group M is a *left A-module* if we are given $A \to \operatorname{End}(M)$.

From now on, *module* means left module unless otherwise indicated. Note that if A is a k-algebra and M is an A-module, then M is also a k-vector space and A acts by k-linear operators.

1.2. (Sept 10) Schur's Lemma. Representations.

1.2.1. Modules [Lor18, §1.2.1].

Definition 1.2.1.1. A morphism $f : M \to M'$ of A-modules is a map of abelian groups such that $f(am) = af(m)$ for all $a \in A$ and $m \in M$.

If A is a k-algebra, then morphisms of A-modules are k-linear transformations.

Lemma 1.2.1.2. *Let A be a ring.*

i. If $\{M_i\}_i$ is a family of A-modules, then

$$\bigoplus_i M_i = \left\{ \sum_i m_i \,\middle|\, m_i \in M_i, \text{ all but finitely many are zero} \right\}$$

is an A-module with action

$$a(\sum_i m_i) = \sum_i am_i.$$

ii. If M is an A-module and $M' \subseteq M$ is a A-submodule, then M/M' is an A-module. The first isomorphism theorem holds for A-modules.

iii. If $f : M \to M''$ is a morphism of A-modules, then the kernel

$$\ker f = \{m \in M \mid f(m) = 0\}$$

and the cokernel

$$\operatorname{coker} f = M'/f(M)$$

are A-modules.

1.2.2. Simple modules. Schur's Lemma.

Definition 1.2.2.1. An A-module M is cyclic if there exists $m \in M$ such that $M = Am$. Then m is a *generator* for M.

Cyclic modules are of the form A/J for a left ideal $J \subseteq A$. The isomorphism is given by $\varphi : A/\ker \varphi \to M$, $\varphi(a) = am$.

Remark 1.2.2.2 (Caroline). Note that in general A/J is not a ring, only a left A-module. A/J is a quotient ring of A if and only if J is a two-sided ideal.

Definition 1.2.2.3. An A-module M is *simple* if $M \neq 0$ and M has no nonzero proper submodules.

Equivalently, M is simple if M is nonzero and has no nonzero proper quotients. Every nonzero element of a simple module is a generator. Simple modules are of the form $A/\mathfrak{m}$ for a maximal left ideal $\mathfrak{m} \subseteq A$: submodules of A/J are exactly J'/J where $J' \supseteq J$ is a left ideal.

Lemma 1.2.2.4 (Schur's Lemma). *Let M and N be simple A-modules. Then a morphism $f : M \to N$ is either zero or an isomorphism.*

Proof. As M is simple, $\ker f = 0$ or $\ker f = M$. In the latter case, $f = 0$, so if f is not zero, then f is injective. As N is simple, $\operatorname{coker} f = 0$ or $\operatorname{coker} f = N$. In the latter case, $f = 0$, so if f is not zero, then f is surjective. Thus f is zero or an isomorphism. $\square$

Corollary 1.2.2.5. *If M is a simple A-module, then $End_A(M)$ is a division ring.*

Corollary 1.2.2.6 (Schur-Dixmier Lemma). *Let A be a nice k-algebra and M a simple A-module. Then* $\operatorname{End}_A(M) = k$ *and $Z(A)$ acts on m by scalars, that is, there exists $\chi_M : Z(A) \to k$ such that $z \cdot m = \chi_M(z)m$ for $z \in Z(A)$ and $m \in M$.*

Proof. By Schur's Lemma, $\operatorname{End}_A(M)$ is a division ring. As M is a simple A-module, M is cyclic, so $\dim \operatorname{End}_A(M) \leq \dim_k M \leq \dim_k A$. As k is central in A, $k \subseteq \operatorname{End}_A(M)$, so $\operatorname{End}_A(M)$ is a nice k-algebra. By the Dixmier Lemma 1.1.2.12, $\operatorname{End}_A(M) = k$. $\square$

Corollary 1.2.2.7 (Weak Nullstellensatz over $\mathbf{C}$). *Let $A = \mathbf{C}[x_1, \ldots, x_n]$. Then the maximal ideals of A are*

$$\mathfrak{m} = (x_1 - z_1, \ldots, x_n - z_n)$$

for $(z_1, \ldots, z_n) \in \mathbf{C}^n$.

Proof. As $\mathbf{C}$ is uncountable and algebraically closed, A is nice. Since A is commutative, $A = Z(A)$. Thus, if M is a simple A-module, there exists $\chi_M : A \to \mathbf{C}$ such that $a \cdot m = \chi_M(a)m$ for all $a \in A$. If x is a generator for M, then $M = A \cdot x = \mathbb{C} \cdot x$, so M is one-dimensional. Thus M is of the stated form, where $z_i = \chi_M(x_i)$. $\square$

1.2.3. Representations and the group algebra.

Let G be a finite group.

Definition 1.2.3.1. If k is a field, a *representation* of G over k is a k-vector space V and a k-linear action $G \times V \to V$ such that $1 \cdot v = v$ for $v \in V$ and $(gh)v = g(hv)$ for all $g, h \in G$ and $v \in V$.

A representation is equivalent to a homomorphism $G \to GL(V)$.

Definition 1.2.3.2. If G is a finite group and k is a field, the *group algebra* kG is the k-algebra with basis G and multiplication

$$\left(\sum_g c_g g\right)\left(\sum_h d_h h\right) = \sum_{g,h} c_g d_h gh.$$

Lemma 1.2.3.3 ([Lor18], §3.1.1). *If A is a k-algebra, then*

$$\{k\text{-algebra homomorphisms } kG \to A\} \cong \{\text{group homomorphisms } G \to A^\times\}.$$

Proof. If $kG \to A$ is a homomorphism, then the image of G is contained in $A^\times$, so by restricting we obtain a morphism $G \to A^\times$. Conversely, a map $\psi : G \to A^\times$ can be extended uniquely to a linear map $kG \to A$. This map will be a ring homomorphism by definition of the product on kG. □

It follows that

$$\begin{aligned}\{\text{representations of } G \text{ on } V\} &\cong \{G \to GL(V)\}| \\ &\cong \{kG \to \mathrm{End}(V)\} \\ &\cong \{\text{left } kG\text{-module structures on } V\}.\end{aligned}$$

Thus, theorems on associative algebras can be applied to the group ring to learn about representations.

Example 1.2.3.4. For all finite groups G, G acts by left multiplication on kG. This representation is the *regular representation.*

Example 1.2.3.5. Let V be a n-dimensional vector space over k with basis $e_1, \dots, e_n$, and let the symmetric group Σ_n act on V by permuting $e_1, \dots, e_n$. V is not simple since $k \cdot (e_1 + \cdots + e_n)$ is invariant under Σ_n, and $W = \{c_1 e_1 + \cdots + c_n e_n \mid \sum_i c_i = 0\}$ is invariant under Σ_n. If $\operatorname{char} k \nmid n$, then

$$V = W \oplus k \cdot (e_1 + \cdots + e_n),$$

but if $\operatorname{char} k \mid n$, then $e_1 + \cdots + e_n \in W$. Then V is not a direct sum of W and k as representations.

Example 1.2.3.6. If V and V' are representations of a finite group G, then:

i. $V \otimes_k V'$ is a representation G with $g(v \otimes v') = gv \otimes gv'$;
ii. $\mathrm{Hom}_k(V, V')$ is a representation of G by $act(g)(f) = gfg^{-1}$.

1.2.4. Maschke's theorem. [Ser78, §1.3]

Theorem 1.2.4.1. *Suppose k is a field and G is a finite group with* $\operatorname{char} k \nmid |G|$. *Let V be a representation of G over k and $U \subseteq V$ be a subrepresentation. Then there exists a subrepresentation $W \subseteq V$ such that $V = U \oplus W$.*

Proof. By extending a basis of U to a basis of V, we may construct a linear map $\tilde{\pi} : V \to U$ such that $\tilde{\pi}(u) = u$ for all $u \in U$. Now let

$$\pi = \frac{1}{|G|} \sum_{g \in G} g \tilde{\pi} g^{-1} : V \to U.$$

The map π is G-linear since

$$h\pi h^{-1} = \frac{1}{|G|}\sum_{g\in G} hg\tilde{\pi}g^{-1}h^{-1} = \frac{1}{|G|}\sum_{g\in G} g\tilde{\pi}g^{-1} = \pi.$$

Further,

$$\pi|_U = \frac{1}{|G|}\sum_{g\in G} g\mathrm{id}_U g^{-1} = \mathrm{id}_U.$$

Thus $W = \ker \pi$ is a complementary subrepresentation to $U \subseteq V$. □

COROLLARY 1.2.4.2. *Let k be a field and G be a finite group with $\operatorname{char} k \nmid |G|$. Then every finite-dimensional representation of G is a direct sum of simple representations.*

PROOF. Induct on the dimension of V. Either V is simple, or V has a proper nonzero submodule W. By Maschke's theorem, $V \cong W \oplus U$ for some submodule $U \subseteq V$. By induction, V is a direct sum of simple representations. □

1.3. (Sept 12) Characters

EXAMPLE 1.3.0.1. Finite abelian group A: assume $n = |A|$ is invertible and k is algebraically closed. Each operator has minimal polynomial dividing $x^n - 1$. As $\frac{d}{dx}x^n - 1 = nx^{n-1}$ and n is invertible, $x^n - 1$ has distinct roots, is diagonalizable, so by homework, the action of A is simultaneously diagonalizable. Thus, a representation of A is a direct sum of one-dimensional representations.

1.3.1. Characters.

In light of Maschke's theorem, to find all the finite-dimensional representations of a finite group, it suffices to find the simple ones. To do so, we will introduce group characters.

From now on, assume k is an algebraically closed field of characteristic zero ($k = \mathbf{C}$).

Recall the trace $\mathrm{tr}(A)$ of a matrix A is the sum of its diagonal entries. The trace satisfies $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ whenever AB and BA are both defined. The trace of a linear endomorphism $f : V \to V$ of a finite-dimensional vector space V is defined to be $\mathrm{tr}(f) = \mathrm{tr}(A)$ whenever A is a matrix for f. As $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(P^{-1}PA) = \mathrm{tr}(A)$, this is well-defined.

More intrinsically, if V and W are vector spaces, then there is a map

$$\begin{aligned} W \otimes V^* &\to \mathrm{Hom}_k(V, W) \\ w \otimes f &\mapsto wf. \end{aligned}$$

This map is an isomorphism if W or V is finite-dimensional. Thus, if V is a finite-dimensional vector space, then $\mathrm{End}_k(V) \cong V \otimes V^*$, and the trace is defined as

$$\mathrm{tr} : \mathrm{End}_k(V) \cong V \otimes V^* \cong V^* \otimes V \to^{ev} k$$

where $ev : V^* \otimes V \to k$ sends $f \otimes v \mapsto f(v)$.

DEFINITION 1.3.1.1. If G is a finite group and V is a finite-dimensional representation of G, the *character* of V is the function

$$\begin{aligned} \chi_V &: G \to k \\ \chi_V(g) &= \mathrm{tr}(act_V(g)). \end{aligned}$$

The character is a *class function* on G: $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $g, h \in G$. You might ask:

> What is the meaning of the trace? Why does it occur in representation theory?

I will attempt to give a few answers over the course of the semester.

Definition 1.3.1.2. If A is a ring and $e \in A$, then e is *idempotent* if $e^2 = e$.

Lemma 1.3.1.3. *Let $f : V \to V$ and $f' : V' \to V'$ be endomorphisms of finite-dimensional vector spaces. Then*

i. $\operatorname{tr}(f \oplus f') = \operatorname{tr}(f) + \operatorname{tr}(f')$;
ii. $\operatorname{tr}(f \otimes f') = \operatorname{tr}(f)\operatorname{tr}(f')$;
iii. $\operatorname{tr}(f^*) = \operatorname{tr}(f)$ *where $f^* : V^* \to V^*$ is the adjoint of f.*
iv. the endomorphism $T \mapsto f'Tf$ of $\operatorname{Hom}(V, V')$ *has trace* $\operatorname{tr}(f)\operatorname{tr}(f')$.
v. if $\lambda : V \to V$ is scalar multiplication by $\lambda \in k$, then $tr(\lambda) = \lambda \dim(V)$;
vi. if $e : V \to V$ is a linear idempotent transformation, then $\operatorname{tr}(e) = rank(e) = \dim im(e)$.

Proof. i. If A and A' are matrices for f and f', then the block diagonal matrix $diag(A, A')$ is a matrix for $f \oplus f'$.
ii. If A and A' are matrices for f and f', then

$$\begin{bmatrix} a_{11}A' & a_{12}A' & \cdots \\ a_{21}A' & a_{22}A' & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

is a matrix for $f \otimes f'$. Thus $\operatorname{tr}(f \otimes f') = \sum_i a_{ii} \operatorname{tr}(A') = \operatorname{tr}(f)\operatorname{tr}(f')$.
iii. If A is a matrix for f, then A^t is a matrix for f^*.
iv. Combine ii., iii., and the observation $\operatorname{Hom}(V, V') \cong V' \otimes V^*$.
v. The trace of the identity matrix is the dimension of the vector space.
vi. If e is idempotent, then $V = \ker(e) \oplus \operatorname{im}(e)$. For each $v \in V$ can be written in the form $v = (1-e)v + ev$, and $e(1-e)v = (e - e^2)v = 0$. Further, if $v \in \ker(e) \cap \operatorname{im}(e)$, then $v = ew$ and $0 = ev = e^2 w = ew = v$.

Now e is the identity on $\operatorname{im}(e)$ and zero on $\ker e$. It follows

$$\operatorname{tr}(e) = \operatorname{tr}(e|_{\ker(e)}) + \operatorname{tr}(e|_{\operatorname{im}(E)}) = \dim \operatorname{im}(e). \qquad \square$$

We are now in [Ser78, §2.3].

Definition 1.3.1.4. If V is a representation of G, the space of *invariants* is

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

Theorem 1.3.1.5. *Let G be a finite group.*

i. Let V be a finite-dimensional representation of G. Then

$$\dim_k V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

ii. Let V and V' be two finite-dimensional representations of G. Then

$$\dim_k \operatorname{Hom}_{kG}(V, V') = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^{-1}) \chi_{V'}(g).$$

PROOF. i. Let $e = \frac{1}{|G|}\sum_{g\in G} g \in kG$. If $h \in G$, then $he = \frac{1}{|G|}\sum_{g\in G} hg = \frac{1}{|G|}\sum_{g\in G} g = e$. Thus $e^2 = e$, so e is idempotent. As $he = e$ for all $h \in G$, the image of e is invariant; if $v \in V^G$ then $ev = \frac{1}{|G|}\sum_{g\in G} v = v$. Thus $\operatorname{im} e = V^G$. By Lemma 1.3.1.3

$$\dim_k V^G = \operatorname{tr}(e) = \frac{1}{|G|}\chi_V(g).$$

ii. Consider the G-representation $\operatorname{Hom}_k(V, V')$ where g acts by $T \mapsto gTg^{-1}$. By Lemma 1.3.1.3, the character of $\operatorname{Hom}_k(V, V')$ is

$$g \mapsto \chi_V(g^{-1})\chi_{V'}(g).$$

The space of invariants $\operatorname{Hom}_k(V, V')^G$ is equal to $\operatorname{Hom}_{kG}(V, V')$: both are the space of linear maps $T : V \to V'$ such that $gT = Tg$ for all $g \in G$. Hence part i. gives

$$\dim \operatorname{Hom}_{kG}(V, V') = \dim \operatorname{Hom}_k(V, V')^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^{-1})\chi_{V'}(g). \qquad \square$$

DEFINITION 1.3.1.6. Define the *product* on functions on G by

$$\langle \chi, \chi' \rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})\chi'(g).$$

Most of the time, we will restrict our attention to class functions, but this is not essential to the definition. The product is a nondegenerate symmetric bilinear form on functions on G, and restricts to a nondegenerate form on the space of class functions. Theorem 1.3.1.5 says

$$\langle \chi_V, \chi_{V'} \rangle = \dim_k \operatorname{Hom}_{kG}(V, V').$$

LEMMA 1.3.1.7 (Schur's lemma for characters). *Let V and V' be representations of G. Then*

$$\langle \chi_V, \chi_{V'} \rangle = \begin{cases} 1 & V \cong V' \\ 0 & V \not\cong V'. \end{cases}$$

Note that we are using $k = \bar{k}$ here.

PROOF. By Schur's lemma, either $\operatorname{Hom}_{kG}(V, V')$ is zero or V is isomorphic to V'. As k is algebraically closed, kG is nice, so the Dixmier-Schur Lemma 1.2.2.6 gives $\operatorname{Hom}_{kG}(V, V') = k$. $\square$

COROLLARY 1.3.1.8. *The set of characters of simple representations is linearly independent.*

COROLLARY 1.3.1.9. *If two finite-dimensional representations V and V' of a finite group G have the same character, then $V \cong V'$.*

PROOF. Let $\{L_i\}_{i\in I}$ be a set of representatives of isomorphism classes of simple representations of G. By Maschke's theorem 1.2.4.2, we may write $V = \bigoplus_i c_i L_i$ and $V' = \bigoplus c_i' L_i$. Then $\chi_V = \chi_{V'}$ implies

$$\sum_i c_i \chi_{L_i} = \sum_i c_i' \chi_{L_i}.$$

Since $\{\chi_{L_i}\}$ is linearly independent, we find $c_i = c_i'$ for all i and thus $V \cong V'$. $\square$

Note that we used that k has characteristic zero to recover the integers c_i and c_i' from their images in k.

1.4. (Sept 17) The number of simples. New representations from old

1.4.1. The number of simple representations. [Ser78, §2.4]

THEOREM 1.4.1.1. *The characters of simple complex representations of G form an orthonormal basis for the space of class functions on G. The number of isomorphism classes of simple representations of G is equal to the number of conjugacy classes.*

PROOF. Let $Cl(G)$ be the space of complex class functions on G. It suffices to show that for $f \in Cl(G)$, if $\langle f, \chi \rangle = 0$ for all simple characters χ, then $f = 0$.

Given such an f, define

$$z = \sum_{g \in G} f(g^{-1}) g \in \mathbf{C}G.$$

Since f is a class function,

$$hzh^{-1} = \sum_{g \in G} f(g^{-1}) hgh^{-1} = \sum_{g \in G} f(h^{-1}gh) g = z,$$

so $z \in Z(\mathbf{C}G)$. By Schur-Dixmier Lemma 1.2.2.6, z acts as a scalar λ_V on an simple representation V. That scalar satisfies

$$\lambda_V \dim_k V = \operatorname{tr}(z : V \to V) = \sum_{g \in G} f(g^{-1}) \chi_V(g) = \langle f, \chi_V \rangle.$$

Thus $\lambda_V = 0$ for all simple V.

Thus z acts as zero on every simple representation. Since the regular representation $\mathbf{C}G$ is a sum of simples, we conclude z acts as zero on the regular representation. However, $z \cdot 1 = z \in \mathbf{C}G$, so z acts as zero on $\mathbf{C}G$ if and only if $z = 0$. Thus $f = 0$, as desired. □

The regular representation played a crucial role in this proof. It's worth recording how the regular representation decomposes:

LEMMA 1.4.1.2. *As a left $\mathbf{C}G$-module,*

$$\mathbf{C}G \cong \bigoplus_{\textit{simple complex } L} L^{\oplus \dim L}.$$

PROOF. If $\mathbf{C}G = \bigoplus_L L^{\oplus c_L}$, then by Schur's lemma, $c_L = \dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{C}G}(\mathbf{C}G, L)$. But

$$\operatorname{Hom}_{\mathbf{C}G}(\mathbf{C}G, L) \cong L$$

as a vector space by $f \mapsto f(1)$. Thus $c_L = \dim L$, as desired. □

COROLLARY 1.4.1.3. *If $\{L_i\}_{i \in I}$ are representatives of the simple modules of G, then*

$$\sum_i \dim(L_i)^2 = |G|.$$

Using Theorem 1.4.1.1 and Corollary 1.4.1.3, you can play "character table sudoku" to compute character tables. Here is a useful lemma for this game:

LEMMA 1.4.1.4. *If V is a finite-dimensional complex representation of finite G, then V is simple if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

PROOF. If $V = \bigoplus_i L_i^{\oplus c_i}$, then

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} c_i c_j \langle \chi_{L_i}, \chi_{L_j} \rangle = \sum_i c_i^2. \qquad \square$$

EXAMPLE 1.4.1.5. Let $G = \Sigma_3$. Note that Σ_3 has three conjugacy classes: $id, (12), (123)$. We know two one-dimensional representations of Σ_3: the trivial and sign characters. The equations $\langle \chi_{triv}, \chi, = \rangle 0$ and $\langle \chi_{alt}, \chi, = \rangle 0$ for the third character give $\chi(1,2) = 0$ and $\chi(1) + 2\chi(1,2,3) = 0$. This determines χ up to a scalar multiple. That scalar is pinned down by the relation $\chi(1)^2 + 1 + 1 = 6$. We have computed the character table.

	id	(12)	(123)
triv	1	1	1
alt	1	-1	1
χ	2	0	-1

What is the associated representation to χ? I claim it is the character of the permutation representation

$$W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}.$$

This can be checked as follows: $W \oplus \mathbf{C}(1,1,1) \cong \mathbf{C}^3$, and $\mathbf{C}^3$ has character

$$\chi_{\mathbf{C}^3}(id) = 3, \chi_{\mathbf{C}^3}(1,2) = 1, \chi_{\mathbf{C}^3}(123) = 0.$$

So $\chi_W = \chi_{\mathbf{C}^3} - \chi_{triv} = \chi$.

1.4.2. Products. [Ser78, §3.2] If V is a representation of G and W is a representation of H, then $V \otimes W$ is a representation of $G \times H$ by

$$(g,h)(v \otimes w) = gv \otimes hw$$

LEMMA 1.4.2.1. *Let G and H be finite groups. If V and W are simple complex representations of G and H, then $V \otimes W$ is a simple complex representation of $G \times H$.*

PROOF. The character of $\chi_{V\otimes W}$ is by Lemma 1.3.1.3

$$\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h).$$

By Lemma 1.4.1.4, it suffies to show $\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = 1$. But

$$\begin{aligned}\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle_{G\times H} &= \frac{1}{|G \times H|} \sum_{g\in G, h\in H} \chi_V(g^{-1})\chi_W(h^{-1})\chi_V(g)\chi_W(h) \\ &= \langle \chi_V, \chi_V \rangle_G \langle \chi_W, \chi_W \rangle_H = 1. \qquad \square\end{aligned}$$

THEOREM 1.4.2.2. *The simple complex representations of $G \times H$ are exactly the tensor products of simples for G and H.*

PROOF. They are simple by Lemma 1.4.2.1. By comparing characters, $V \otimes W \cong V' \otimes W'$ over $G \times H$ if and only if $V \cong V'$ and $W \cong W'$. The number of such representations is the product of the number of conjugacy classes of G and of H, which is the number of conjugacy classes of $G \times H$. By Theorem 1.4.1.1, we have found all of the irreducible complex representations. $\square$

1.4.3. Restriction, Induction, Coinduction. In this section, k is an arbitrary field. Let G be a group and $H \subseteq G$ be a subgroup. Then there is the functor

$$\operatorname{Res}_H^G : \operatorname{Rep}(G) \to \operatorname{Rep}(H)$$

which sends a G-representation V to V, viewed as a representation of H. This is a functor since if $f : V \to W$ is G-linear, it is also H-linear when we restrict to H.

The functor Res_H^G has a left adjoint, *induction*

$$\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G),$$

in the sense that

$$\operatorname{Hom}_{kG}(\operatorname{Ind}_H^G V, W) \cong \operatorname{Hom}_{kH}(V, \operatorname{Res}_H^G W),$$

and this isomorphism is natural in V and W.

The construction of Ind_H^G involves the tensor product, not over k, but over other rings. If $A \to B$ is a morphism of rings and M is a left A-module, then we can form

$$B \otimes_A M = B \otimes M / \{ba \otimes m - b \otimes am \mid a \in A, b \in B, m \in M.\}.$$

Note that $B \otimes_A M$ is a B-module by left multiplication by B (the left action doesn't know what the right A-action is doing).

Lemma 1.4.3.1 ([Lor18], 1.2.2). *If $A \to B$ is a ring homomorphism, M is an A-module, and N is a B-module, then*

$$\operatorname{Hom}_B(B \otimes_A M, N) \cong \operatorname{Hom}_A(M, N)$$

where we view N as an A-module via $A \to B$.

Proof. For

$$\begin{aligned}\operatorname{Hom}_B(B \otimes_A M, N) = \{f : B \times M \to N \mid & f \text{ is bilinear}, \\ & f(ba, m) = f(b, am) \text{ for all } a \in A, \\ & \text{and } f(bx, m) = bf(x, m).\}\end{aligned}$$

Such a map $f : B \times M \to N$ is determined by $f(1, -) : M \to N$, and $f : B \times M \to N$ is balanced with respect to A if and only if $f(1, -)$ is A-linear. □

Definition 1.4.3.2. If V is a G-representation over k, then the induced representation $\operatorname{Ind}_H^G V = kG \otimes_{kH} V$.

Note that $kG = \bigoplus_{\sigma \in G/H} \sigma kH$ as a right kH-module, so as a vector space,

$$\operatorname{Ind}_H^G V = \bigoplus_{\sigma \in G/H} \sigma kH \otimes_{kH} V = \bigoplus_{\sigma \in G/H} \sigma V.$$

In particular, $\dim \operatorname{Ind}_H^G V = [G : H] \dim V$ when one side of the equality is finite.

The functor Res_H^G also has a right adjoint

$$\operatorname{Coind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G),$$

given by

$$\operatorname{Coind}_H^G(W) = \operatorname{Hom}_{kH}(kG, W),$$

where G acts on $\operatorname{Hom}_{kH}(kG, W)$ as follows: if $f : kG \to W$, then

$$(g \cdot f)(x) = f(xg).$$

Exercise!: check that this defines an action of G.

Lemma 1.4.3.3 ([Lor18], 1.2.2). *If $A \to B$ is a ring homomorphism, M is an A-module, and N is a B-module, then*

$$\operatorname{Hom}_A(N, M) \cong \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, M)),$$

where $\operatorname{Hom}_A(B, M)$ is viewed as a left B-module by $b \cdot f(b') = f(b'b)$.

Proof. Skipped. The map sends $h : \operatorname{Hom}_A(N, M)$ to $\tilde{h} : N \to \operatorname{Hom}_A(B, M)$ by $\tilde{h}(n)(b) = bh(n)$. ☐

Since $kG = \bigoplus_{\sigma \in H\backslash G} kH\sigma$ as a left kH-module,

$$\operatorname{Coind}_H^G(W) \cong \prod_{\sigma \in H\backslash G} W\sigma$$

as a vector space.

1.5. (Sept 19) Mackey theorem

1.5.1. Comparing induction and coinduction.

Theorem 1.5.1.1 ([Lor18], Proposition 3.4). *If $[G : H]$ is finite, then there is a natural isomorphism $\operatorname{Ind}_H^G W \cong \operatorname{Coind}_H^G W$ for $W \in \operatorname{Rep}(H)$.*

Proof. Let $\pi : kG \to kH$ by

$$\pi(g) = \begin{cases} 0 & g \notin H \\ g & g \in H \end{cases}.$$

Then $\pi(hg) = h\pi(g)$ and $\pi(gh) = \pi(g)h$ for $h \in H$ and $g \in G$. Thus π defines a map

$$W = \operatorname{Hom}_{kH}(kH, W) \to \operatorname{Hom}_{kH}(kG, W) = \operatorname{Res}_H^G \operatorname{Coind}_H^G W.$$

By adjunction we obtain

$$\phi : \operatorname{Ind}_H^G W \to \operatorname{Coind}_H^G W.$$

Check that at the level of vector spaces, ϕ is given by

$$\bigoplus_{\sigma \in G/H} \sigma W \to \prod_{\sigma \in H\backslash G} W\sigma$$

by sending σW isomorphically to $W\sigma^{-1}$. ☐

In the context of finite-dimensional complex representations of finite groups, we will thus refer to just the induction functor, and use that it is both left and right adjoint.

1.5.2. Character formula for the induced representaiton.

If χ is a character of H, let $\operatorname{Ind}_H^G \chi$ be the character of the induced representation.

Lemma 1.5.2.1. *If $\tilde{\sigma}$ is a coset representative for each $\sigma \in G/H$,*

$$\operatorname{Ind}_H^G \chi(g) = \sum_{\sigma \in G/H} \begin{cases} \chi(\tilde{\sigma}^{-1} g \tilde{\sigma}) & \tilde{\sigma}^{-1} g \tilde{\sigma} \in H \\ 0 & \tilde{\sigma}^{-1} g \tilde{\sigma} \notin H. \end{cases}$$

Note that each term in the sum above does not depend on the choice of coset representative.

Proof. Let W be a representation with character χ. Then $\operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} \sigma W$. An element $g \in G$ takes σW to $g\sigma W$. Thus a matrix for g acting on $\operatorname{Ind}_H^G W$ will be a block matrix, with blocks corresponding to the action of g on G/H. The diagonal blocks are those σ where $g\sigma = \sigma$, in which case g acts by

$$g\tilde{\sigma}w = \tilde{\sigma}(\tilde{\sigma}^{-1}g\tilde{\sigma})w$$

when $\tilde{\sigma}$ is a coset representative for σ. Thus

$$tr(g; \operatorname{Ind}_H^G W) = \sum_{\sigma \in G/H g\sigma = \sigma} tr(g; \sigma W) = \sum_{\sigma \in G/H g\sigma = \sigma} \chi(\tilde{\sigma}^{-1}g\tilde{\sigma}). \qquad \square$$

Theorem 1.5.2.2 (Frobenius reciprocity). *For two complex characters χ and ψ,*

$$\langle \operatorname{Ind}_H^G \chi, \psi \rangle_G = \langle \chi, \operatorname{Res}_H^G \psi \rangle_H.$$

Proof. Let χ be the character of W and ψ be the character of V. By Theorem 1.3.1.5, these products compute the dimension of

$$\operatorname{Hom}_{\mathbf{C}G}(\operatorname{Ind}_H^G W, V) \cong \operatorname{Hom}_{\mathbf{C}H}(W, \operatorname{Res}_H^G V). \qquad \square$$

1.5.3. Examples of induced representation.

Example 1.5.3.1. If X is a set with G-action, define kX to be the free vector space on X. Then kX is a G-representation given by linearizing the action of G on X.

If $X \cong G/H$ is transitive, then

$$k(G/H) \cong kG \otimes_{kH} k = \operatorname{Ind}_H^G k,$$

where we view k as a trivial H-representation. Thus every linearized set representation is a direct sum of induced representations.

Example 1.5.3.2. Σ_3 acting on $\mathbf{C}^3$ is induced from the trivial representation of $\Sigma_2 \subseteq \Sigma_3$, the stabilizer of $(0, 0, 1)$.

1.5.4. Mackey theorem. [Ser78, §7.3]. The Mackey theorem describes the effect of inducing up, then restricting down.

Theorem 1.5.4.1 (Mackey). *Let G be a group and H, K be two subgroups of G. Let W be a representation of H over a field k and $\rho : H \to GL(W)$ the action. Then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{s \in K\backslash G/H} \operatorname{Ind}_{H_s}^K W_s,$$

where $H_s = K \cap \tilde{s}H\tilde{s}^{-1}$ for some fixed representative $\tilde{s}$ of s, and W_s is the representation of H_s defined by $\rho^{\tilde{s}}(h) = \rho(\tilde{s}^{-1}h\tilde{s})$.

Proof. Note $\operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} \sigma W$, and if $k \in K$, then $k \cdot \sigma W \subseteq (k \cdot \sigma)W$. Thus, if $s \in K\backslash G/H$, let

$$V(s) = \bigoplus_{\sigma \in G/H, \sigma \subseteq s} \sigma W;$$

then $KV(s) \subseteq V(s)$, so

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{s \in K\backslash G/H} V(s).$$

Now let $\tilde{s} \in G$ be a representative for the double coset s. If $h \in H_s$, then $\tilde{s}^{-1}h\tilde{s} \in H$ by definition of H_s. Thus $h\tilde{s}W = \tilde{s}(\tilde{s}^{-1}h\tilde{s})W \subseteq \tilde{s}W$. Note $\tilde{s}W = W_s$ as a H_s-representation by definition. There is a bijection $K/H_s \cong \{\sigma \in G/H \mid \sigma \subseteq s\}$ of left K-sets sending 1 to $K\tilde{s}H$. So $V(s) = \bigoplus_{\sigma\subseteq s} \sigma W = \bigoplus_{x\in K/H_s} x\tilde{s}W$. This is the induced representation $\operatorname{Ind}_{H_s}^{K} W_s$. The proof is complete. □

Here is an example application of the Mackey theorem.

Corollary 1.5.4.2. *If $H \subseteq G$ is a subgroup of a finite group, then for W a complex f.d. representation of G, $\operatorname{Ind}_H^G W$ is irreducible if and only if*

i. W is irreducible, and
ii. for all $s \in G - H$, W_s and $\operatorname{Res}_{H_s} W$ are disjoint representations of $H_s = H \cap sHs^{-1}$.

Proof. The induced representation is irreducible when

$$\operatorname{Hom}_{\mathbf{C}G}(\operatorname{Ind}_H^G W, \operatorname{Ind}_H^G W) = \mathbf{C}.$$

But

$$\operatorname{Hom}_{\mathbf{C}G}(\operatorname{Ind}_H^G W, \operatorname{Ind}_H^G W) = \operatorname{Hom}_{\mathbf{C}H}(W, \operatorname{Res}_H^G \operatorname{Ind}_H^G W) = \operatorname{Hom}_{\mathbf{C}H}(W, \bigoplus_{\sigma\in G/H} \operatorname{Ind}_{H_s}^H W_s).$$

Thus we require $\operatorname{Hom}_{\mathbf{C}H}(W, W) = \mathbf{C}$ and for $s \in G - H$,

$$0 = \operatorname{Hom}_{\mathbf{C}H}(W, \operatorname{Ind}_{H_s}^H W_s) = \operatorname{Hom}_{\mathbf{C}H_s}(\operatorname{Res}_{H_s} W, W_s).$$ □

You can use this criterion to analyze representations of the semidirect product of groups. See for example [Ser78, §8.2].

1.6. (Sept 24) Density theorem

1.6.1. Decomposition of the regular representation. Density theorem. Note that $\mathbf{C}G$ actually has two actions of G: left multiplication by G and right multiplication by G. Thus $\mathbf{C}G$ is a representation not of G, but of $G \times G$ with action $(g, h) \cdot x = gxh^{-1}$. How does $\mathbf{C}G$ decompose into representations of $G \times G$. Furthermore, $\mathbf{C}G$ is not just a representation but an an algebra. What is the algebra structure?

Theorem 1.6.1.1 (Density theorem. [Ser78], §6.2). *Let $\{L_i\}$ be a set of isomorphism classes of simple complex representations of G. Then the action map*

$$act : \mathbf{C}G \to \prod_i \operatorname{End}_{\mathbf{C}}(L_i)$$

is an isomorphism of $\mathbf{C}$-algebras.

Corollary 1.6.1.2 (Peter-Weyl theorem). *As representations of $G \times G$,*

$$\mathbf{C}G \cong \prod_{simple\ L} L \otimes L^*$$

and

$$\{functions\ G \to \mathbf{C}\} \cong \bigoplus_{simple\ L} L^* \otimes L.$$

Note that $L \otimes L^*$ is a simple representation of $G \times G$ by Lemma 1.4.2.1.

To prove the density theorem, we need to construct certain functions on G.

Definition 1.6.1.3. A *matrix entry* of a representation V over k is a function $m_{f,v} : G \to k$ depending on $f \in V^*$ and $v \in V$, defined by

$$m_{f,v}(g) = f(gv).$$

Lemma 1.6.1.4. *Let V and V' be simple representations of G over a characteristic zero field k. Let $v \in V$, $f \in V^*, v' \in V', f' \in (V')^*$. Then*

$$\langle m_{f,v}, m_{f',v'} \rangle = \begin{cases} 0 & V \ncong V' \\ \frac{1}{\dim V} f'(v) f(v') & V = V'. \end{cases}$$

Proof. Observe

$$\langle m_{f,v}, m_{f',v'} \rangle = \frac{1}{|G|} \sum_{g \in G} f(gv) f'(g^{-1}v') = f\left(\left(\frac{1}{|G|} \sum_{g \in G} gvf'g^{-1} \right) v' \right).$$

Now $y = \frac{1}{|G|} gvf'g^{-1}$ is a G-invariant operator $V' \to V$. If $V \ncong V'$, then $y = 0$, which shows the first equation of the Lemma. Thus we can assume $V = V'$. Thus y is a scalar operator $V \to V$ acting by $\frac{1}{\dim V} \operatorname{tr}(y)$. But $\operatorname{tr}(y) = \operatorname{tr}(vf') = f'(v)$. Hence

$$\langle m_{f,v}, m_{f',v'} \rangle = f(yv') = \frac{1}{\dim V} f'(v) f(v'). \qquad \square$$

If one takes v ranging through a basis of V and f ranging through a dual basis, one finds that these matrix entries are orthogonal.

Proof of Theorem 1.6.1.1. To show *act* is surjective, it suffices to show that $act^* : \bigoplus_i \operatorname{End}_{\mathbf{C}}(L_i)^* \to \mathbf{C}G^*$ is injective. However,

$$\operatorname{End}_{\mathbf{C}}(L)^* = (L \otimes L^*)^* = L^* \otimes L.$$

The pure tensor $f \otimes v \in L^* \otimes L$ corresponds to the linear function $T \mapsto fTv$ in $\operatorname{End}_{\mathbf{C}}(L)^*$. Thus $\operatorname{End}_{\mathbf{C}}(L_i)^*$ has a basis of the form $T \mapsto fTv$ where $f \in L_i^*$ and $v \in L_i$ run through bases. Now $act^*(T \mapsto fTv) = m_{f,V}$ is the matrix entry for (f, v). By Lemma 1.6.1.4, the images of $\operatorname{End}_{\mathbf{C}}(L_i)^*$ are orthogonal, so it suffices to show $act^* : \operatorname{End}_{\mathbf{C}}(L)^* \to \mathbf{C}G^*$ is injective for all simple representations L.

If we pick a basis $\{e_1, \dots, e_n\}$ for L and a dual basis $\{f_1, \dots, e_n\}$ for L^*, and set $m_{i,j} = m_{f_i, e_j}$,

$$\langle m_{i,j}, m_{k,\ell} \rangle = \begin{cases} \frac{1}{\dim L} & i = \ell \text{ and } j = k \\ 0 & \text{else.} \end{cases}$$

Thus $\{m_{i,j}\}$ is linearly independent in $\mathbf{C}G^*$. But $\{m_{i,j}\}$ is the image of the basis $f_i \otimes e_j \in L^* \otimes L = (\operatorname{End}_{\mathbf{C}}(L))^*$. Thus act^* is injective.

By Corollary 1.4.1.3, $\dim \mathbf{C}G = \sum_i \dim_{\mathbf{C}}(L_i)^2$, so the surjective map $act : \mathbf{C}G \to \bigoplus_i \operatorname{End}_{\mathbf{C}}(L_i)$ is an isomorphism. $\square$

1.6.2. Central idempotents. The isomorphism $\mathbf{C}G \cong \prod_L \operatorname{End}_{\mathbf{C}}(L)$ implies that $Z(\mathbf{C}G) = \prod_L \mathbf{C}$. Thus, for each simple L, there exists $e_L \in Z(\mathbf{C}G)$ such that e_L acts as the identity on L and zero on simple $L' \ncong L$.

Lemma 1.6.2.1.

$$e_L = \frac{\dim L}{|G|} \sum_{g \in G} \chi_L(g^{-1}) g \in Z(\mathbf{C}G)$$

acts as the identity on L and as zero on $L' \ncong L$.

Proof. Let $z_L = \frac{\dim L}{|G|}\sum_{g\in G}\chi_L(g^{-1})g$. Since χ_L is a class function, $z_L \in Z(\mathbf{C}G)$. Then

$$\operatorname{tr}(z_L; L') = \frac{\dim L}{|G|}\sum_{g\in G}\chi_L(g^{-1})\chi_{L'}(g) = \dim L\langle \chi_L, \chi_{L'}\rangle.$$

By Schur-Dixmier lemma 1.2.2.6, z_L acts as the scalar $(\dim L')^{-1}\operatorname{tr}(z_L; L')$ on L'. Thus z_L acts by 1 on L and 0 on $L' \ncong L$. $\square$

1.6.3. Isotypic components.

Definition 1.6.3.1. If G is a finite group, V is a complex representation of G, and L is a simple complex representation of G, the *L-isotypic component of V*, written V_L, is the sum of all subrepresentations of V isomorphic to L.

The isotypic component V_L is entirely canonical. If we have already decomposed V into simple representations, then V_L is just the direct sum of those simples isomorphic to L. Also, $V_L = e_L V$.

1.7. (Sept 26) Double density theorem

1.7.1. Remarks on density theorem without coordinates. If V and W are finite-dimensional vector spaces over a field k, then

$$\operatorname{Hom}(V,W)^* = (W\otimes V^*)^* = W^*\otimes V \cong V\otimes W^* = \operatorname{Hom}(W,V).$$

Thus there is a duality pairing

$$\operatorname{Hom}(V,W)\times \operatorname{Hom}(W,V)\to k.$$

This pairing is $A, B \mapsto \operatorname{tr}(AB)$. (This is a nice exercise in the coordinate-free definition of trace and matrix multiplication.)

Thus, if L is a simple complex G-representation and $A \in (\operatorname{End}_{\mathbf{C}}(L))^* = \operatorname{End}_{\mathbf{C}}(L)$, the resulting $act^*A \in (\mathbf{C}G)^*$ is the function $g \mapsto \operatorname{tr}(gA; L)$. If $A = vf$ corresponds to the rank one tensor $f\otimes v \in L^*\otimes L$, then

$$\operatorname{tr}(gvf) = \operatorname{tr}(fgv) = f(gv) = m_{f,v}(g)$$

by the cyclic property of the trace, so this definition extends our definition of matrix entries earlier.

The orthogonality Lemma 1.6.1.4 may be generalized as follows:

Lemma 1.7.1.1. *If L is a simple G-representation, then for $A, B \in \operatorname{End}_{\mathbf{C}}(L)$,*

$$\langle act^*A, act^*B\rangle = \frac{1}{\dim L}\operatorname{tr}(AB).$$

Thus $act^* : \operatorname{End}_{\mathbf{C}}(L)^* \to (\mathbf{C}G)^*$ is injective because it is a scalar multiple of an isometry. This allows you to finish the proof of the density theorem 1.6.1.1 without coordinates.

1.7.2. Double density theorem.

Theorem 1.7.2.1 (Double density). *Let G be a finite group, V a finite-dimensional complex representation of G, and $A = \operatorname{End}_{\mathbf{C}G}(V) \subseteq \operatorname{End}_{\mathbf{C}}(V)$. Then:*

i. there is a natural direct sum decomposition

$$V = \bigoplus_{L \text{ simple } G\text{-rep}} \mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L$$

of V into simple $A \otimes_{\mathbf{C}} \mathbf{C}G$-modules.

ii. The action map

$$A \to \prod_L \mathrm{End}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}G}(L, V))$$

is an isomorphism.

iii. (Double centralizer property) $\mathrm{End}_A(V)$ is equal to the image of $\mathbf{C}G$ in $\mathrm{End}_{\mathbf{C}}(V)$.

There is always a map from $\mathbf{C}G$ into the centralizer of the centralizer of $\mathbf{C}G$: G commutes with all operators which commute with G. Thus we have a map $\mathbf{C}G \to \mathrm{End}_A(V)$, and the last part of the theorem says this map is surjective.

LEMMA 1.7.2.2. *Let V and W be finite-dimensional vector spaces over a field k, and let $B = \mathrm{End}_k(V)$. Then B acts on $V \otimes W$, and*

$$\mathrm{End}_B(V \otimes W) = \mathrm{End}_k(W).$$

PROOF. First observe that A and B commute, for if $a \in A$ and $b \in B$,

$$ab(v \otimes w) = bv \otimes aw = ba(v \otimes w).$$

Thus we find a map $\mathrm{End}_k(W) \to \mathrm{End}_B(V \otimes W)$. To check that this map is an isomorphism, pick a basis $\{e_1, \dots, e_n\}$ for W. Then $\mathrm{End}_k(W)$ is identified with the matrix ring $M_n(k)$. Further,

$$V \otimes W = V \otimes \bigoplus_{i=1}^{n} k e_i = \bigoplus_{i=1}^{n} V.$$

Hence

$$\mathrm{End}_B(V \otimes W) = \mathrm{Hom}_B(\bigoplus_{i=1}^{n} V, \bigoplus_{j=1}^{n} V) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathrm{Hom}_B(V, V) = M_n(k)$$

since $\mathrm{Hom}_B(V, V) = k$. Under these identifications the map $\mathrm{End}_k(W) \to \mathrm{End}_B(V \otimes W)$ is identified with the identity $M_n(k) \to M_n(k)$. Thus this map is an isomorphism. □

PROOF. Note that $\mathrm{Hom}_{\mathbf{C}G}(L, V)$ is an A-module as follows: if $f : L \to V$ is G-linear and $a \in A$, then $af : L \to V$ is also G-linear since $gaf = agf = afg$. First we prove that V decomposes as a direct sum above. Consider the map

$$\begin{aligned} ev_L : \mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L &\to V \\ f \otimes \ell &\mapsto f(\ell). \end{aligned}$$

Note that ev_L is $A \otimes \mathbf{C}G$-linear, since

$$ev_L(a \otimes g)(f \otimes \ell) = af(g\ell) = agf(\ell) = (a \otimes g)ev_L(f \otimes \ell).$$

Taking the direct sum over isomorphism classes of simple L gives a map

$$\bigoplus_L ev_L : \bigoplus_L \mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L \to V$$

of $A \otimes \mathbf{C}G$-modules. I claim $\bigoplus_L ev_L$ is an isomorphism. Since V is a direct sum of simples and both sides are linear with respect to direct sum, it suffices to show that this map is an isomorphism for simple $L' \cong V$. Then $\mathrm{Hom}_{\mathbf{C}G}(L, L')$ is either 0 or $\mathbf{C}$ by Schur's Lemma 1.2.2.6, so our map is

$$\bigoplus_L \mathrm{Hom}_{\mathbf{C}G}(L, L') \otimes L = \mathrm{Hom}_{\mathbf{C}G}(L, L) \otimes L = \mathbf{C} \otimes L = L \to L.$$

The map is an isomorphism.

Thus

$$V = \bigoplus_{L \text{ simple } G\text{-rep}} \mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L$$

as $A \otimes \mathbf{C}G$-modules.

Now we prove ii. Since A acts on $\mathrm{Hom}_{\mathbf{C}G}(L, V)$ for all L, we obtain an action map

$$A \to \prod_L \mathrm{End}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}G}(L, V)).$$

We want to show that this action map is an isomorphism. By the Density Theorem 1.6.1.1, $\mathbf{C}G = \prod_L \mathrm{End}_{\mathbf{C}}(L)$ as $\mathbf{C}$-algebras. As $\mathrm{Hom}_{\mathbf{C}G}(L, L') = 0$ for disinct simple L and L',

$$A = \mathrm{End}_{\mathbf{C}G}(V) = \bigoplus_L \mathrm{End}_{\mathbf{C}G}(\mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L). \tag{1}$$

By the density theorem, $\mathbf{C}G \to \mathrm{End}_{\mathbf{C}} L$ is surjective. Thus by Lemma 1.7.2.2,

$$\mathrm{End}_{\mathbf{C}G}(\mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L) = \mathrm{End}_{\mathrm{End}_{\mathbf{C}} L}(\mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L) = \mathrm{End}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}G}(L, V)).$$

Thus the isomorphism (1) is the action map $A \to \prod_L \mathrm{End}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}G}(L, V))$. We have proved ii. Further we find $\mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L$ is a simple $A \otimes \mathbf{C}G$-module, finishing i.

Finally, since A is the product of matrix algebras, we find symmetrically

$$\mathrm{End}_A(V) = \bigoplus_{L \text{ appearing in } V} \mathrm{End}_A(\mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L) = \bigoplus_{L \text{ appearing in } V} \mathrm{End}_{\mathbf{C}}(L).$$

By the Density Theorem 1.6.1.1, $\mathbf{C}G$ surjects onto $\prod_{L \text{ appearing in } V} \mathrm{End}_{\mathbf{C}}(L)$, proving iii. ☐

REMARK 1.7.2.3. We have found a coordinate-independent formula for the isotypic component of V:

$$V_L \cong \mathrm{Hom}_{\mathbf{C}G}(L, V) \otimes L.$$

1.7.3. The Symmetric Group.

DEFINITION 1.7.3.1. The symmetric group Σ_n is the group of permutations of n letters $\{1, 2, \ldots, n\}$.

The inclusion $\{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n\}$ gives an injective homomorphism $\Sigma_{n-1} \to \Sigma_n$. The image of this homomorphism is the stabilizer of n. Our approach to the complex representation theory of Σ_n is to consider the chain of subgroups

$$\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots.$$

Given this chain, we can study a representation of Σ_n by studying how it restricts to these subgroups. Dually, we want to build up representations of Σ_n by studying induced modules from these subgroups.

1.8. (Oct 01) Simple branching and the symmetric group

1.8.1. Simple branching. We are now following [VO04, §1].

Definition 1.8.1.1. A pair (G, H) of a group G and a subgroup $H \subseteq G$ has *simple branching* if for all simple representations V of G, the restriction $\operatorname{Res}_H^G V$ is a direct sum of simple H-representations with multiplicity one.

A pair (G, H) having simple branching is also known in the literature as a *strong Gelfand pair.*

Example 1.8.1.2. Consider $\Sigma_2 \subseteq \Sigma_3$. The restriction of one-dimensional representatioins are one-dimensional and so trivially have multiplicity one. The other character χ has $\chi|_{\Sigma_2} = \chi_1 + \chi_{alt}$. Thus $\Sigma_2 \subseteq \Sigma_3$ has simple branching.

Example 1.8.1.3. When does $1 \subseteq G$ have simple branching? There is only one simple representation of 1, so $(G, 1)$ has simple branching if and only if every simple representation has dimension 1. This occurs only when G is abelian.

Theorem 1.8.1.4 ([VO04], Proposition 1.4). *(G, H) has simple branching if and only if the centralizer*

$$Z(G, H) = \{x \in \mathbf{C}G \mid hx = xh \text{ for all } h \in H\}$$

is commutative.

Proof. By Theorem 1.6.1.1, $\mathbf{C}G = \prod_L \operatorname{End}_{\mathbf{C}}(L)$ over simples L. Thus

$$Z(G, H) = \prod_L \operatorname{End}_{\mathbf{C}H}(\operatorname{Res}_H^G L).$$

By Theorem 1.7.2.1,

$$\operatorname{End}_{\mathbf{C}H}(\operatorname{Res}_H^G L) = \prod_{L' \text{simple for } H} \operatorname{End}_{\mathbf{C}}(\operatorname{Hom}_{\mathbf{C}H}(L', \operatorname{Res}_H^G L)),$$

a product of matrix algebras of size $c \times c$ where c ranges over $\dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{C}H}(L', \operatorname{Res}_H^G L)$. Thus $Z(G, H)$ is commutative if and only if $\dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{C}H}(L', \operatorname{Res}_H^G L) \leq 1$ for all simple H-representations L' and all simple G-representations L, i.e. if and only if (G, H) has simple branching. □

Lemma 1.8.1.5 ([VO04], §2). *If (G, H) is a pair such that each element $g \in G$ is conjugate to its inverse by an element $h \in H$, then (G, H) has simple branching.*

Proof. Let $S : \mathbf{C}G \to \mathbf{C}G$ be the antipode map

$$g \mapsto g^{-1}.$$

Since $(gh)^{-1} = h^{-1}g^{-1}$ for $g, h \in G$, the map S is a ring anti-automorphism, i.e. $S(xy) = S(y)S(x)$. Since S preserves $\mathbf{C}H$, it also preserves its centralizer $Z(G, H)$. Now let $x = \sum_i c_g g$ be in $Z(G, H)$. Suppose that $hgh^{-1} = g^{-1}$ for $h \in H$. Then $hg^{-1}h^{-1} = g$, and conjugation by h permutes the other elements appearing in x. But $hxh^{-1} = x$, so the coefficients of g and g_{-1} in x are equal: $c_g = c_{g^{-1}}$. It follows that $S(x) = x$ for $x \in Z(G, H)$. Since S is a ring anti-automorphism and the identity, we find $xy = S(xy) = S(y)S(x) = yx$ for all $x, y \in Z(G, H)$, that is, $Z(G, H)$ is commutative. Now apply Theorem 1.8.1.4. □

Theorem 1.8.1.6. *(Σ_n, Σ_{n-1}) has simple branching for all $n > 1$.*

PROOF. Two elements of Σ_n are conjugate if they have the same cycle type. Note n appears in a cycle of the same length in σ and σ^{-1}. Thus we can pick an element conjugating σ to σ^{-1} which fixes n, that is, lies in Σ_{n-1}.

Thus Lemma 1.8.1.5 applies, and (Σ_n, Σ_{n-1}) has simple branching. □

DEFINITION 1.8.1.7. The nth *Gelfand-Tsetlin algebra* is

$$GZ(n) = \langle Z(\mathbf{C}\Sigma_i) \mid i \leq n \rangle.$$

THEOREM 1.8.1.8. *$GZ(n)$ acts simultaneously diagonalizably on every representation of Σ_n. Each simultaneous eigenspace is one-dimensional.*

PROOF. Since (Σ_i, Σ_{i-1}) has simple branching for all $i > 1$, the restriction of a simple $\mathbf{C}\Sigma_i$-module V decomposes canonically into simple modules. So if V is a simple Σ_n-representation, applying this decomposition canonically gives a decomposition

$$V = \bigoplus_T \mathbf{C}_T$$

into simple Σ_1-representations $\mathbf{C}_T$ indexed by tuples $T = (L_1, L_2, \dots L_{n-1})$ where L_{i-1} is a simple constituent of $\mathrm{Res}^{\Sigma_i}_{\Sigma_{i-1}} L_i$ for all $i \leq n$ (taking the convention $L_n = V$). Since $\Sigma_1 = \{1\}$, $\mathbf{C}_T$ is a one-dimensional vector space.

As $Z(\mathbf{C}\Sigma_i)$ acts by scalars on every simple L_i-module, we see $\mathbf{C}_T$ is stable under $GZ(n)$. Further, if $T \neq T'$, then for some i, the simple constituents for Σ_i in T and T' are different. Thus $Z(\mathbf{C}\Sigma_i)$ acts by different scalars on $\mathbf{C}_T$ and $\mathbf{C}_{T'}$. Thus $\{\mathbf{C}_T\}$ is the set of simultaneous eigenspaces, all one-dimensional. □

DEFINITION 1.8.1.9. A *Gelfand-Tsetlin basis* is a basis of simultaneous eigenvectors for $GZ(n)$.

LEMMA 1.8.1.10. *Let V be a simple representation of Σ_n. If $\langle -, - \rangle$ is an invariant Hermitian form on V, then a Gelfand-Tsetlin basis for V is orthogonal.*

PROOF. Suppose V is a simple Σ_n-representation. By simple branching (Theorem 1.8.1.6), $\mathrm{Res}_{\Sigma_{n-1}} V = \bigoplus_i L_i$ where L_i are distinct simple representations of Σ_{n-1}. Then $L_i^{\perp}$, being a subrepresentation of V disjoint from L_i, must be $\bigoplus_{j \neq i} L_j$. Thus distinct summands of $\mathrm{Res}_{\Sigma_{n-1}} V$ are orthogonal. Since GT bases for V restrict to GT bases for the summands of the restrction, we conclude by induction that a Gelfand-Tsetlin basis for V is orthogonal. □

Our classification of simple representations of Σ_n will involve analyzing the action of $GZ(n)$ on this basis, which in particular specifies the branching rule for (Σ_n, Σ_{n-1}).

1.8.2. YJM elements.

DEFINITION 1.8.2.1. The nth *Young-Jucys-Murphy* element is

$$X_n = (1n) + (2n) + \cdots + (n-1, n) \in \mathbf{C}\Sigma_n.$$

Note $X_1 = 0$ and X_n is the difference of the sum of transpositions in Σ_{n-1} and Σ_n. Thus $X_n \in Z\mathbf{C}\Sigma_{n-1} + Z\mathbf{C}\Sigma_n \subseteq GZ(n)$.

LEMMA 1.8.2.2.

$$Z(\Sigma_n, \Sigma_{n-1}) = \langle Z(\mathbf{C}\Sigma_{n-1}), X_n \rangle.$$

Proof. Certainly $\langle \mathbf{Z}(\mathbf{C}\Sigma_{n-1}), X_n\rangle \subseteq Z(\Sigma_n, \Sigma_{n-1})$. Now note that the centralizer has a basis of indicator sums of *marked cycle types*: these are cycle types where we remember where n is. If ν is a marked cycle type, let $\ell(\nu)$ be the total lengths of all nontrivial cycles, and let

$$t_\nu = \sum_{\sigma \in \nu} \sigma \in Z(\Sigma_n, \Sigma_{n-1}).$$

Then $\{t_\nu\}_{\nu \text{ marked cycle type}}$ is a basis for $Z(\Sigma_n, \Sigma_{n-1})$.

We show $t_\nu \in \langle Z(\mathbf{C}\Sigma_{n-1}), X_n\rangle$ by induction on $\ell(\nu)$. The base case is $\ell(\nu) = 0$, when $t_\nu = 1 \in Z(\mathbf{C}\Sigma_{n-1})$. Now let ν be given. If $\nu = \nu' \sqcup \nu''$ has two disjoint cycles, then $t_{\nu'}, t_{\nu''} \in \langle Z(\mathbf{C}\Sigma_{n-1}), X_n\rangle$ by induction. Now

$$t_{\nu'} t_{\nu''} = c t_\nu + \sum (\text{ smaller } \ell\text{'s})$$

for some positive integer c, so by induction $t_\nu \in \langle Z\mathbf{C}\Sigma_{n-1}, X_n\rangle$.

We are left to consider when ν is a single cycle. If the cycle ν does not contain n, then $t_\nu \in Z(\mathbf{C}\Sigma_{n-1})$. Now suppose the cycle in ν contains n. Consider a cycle $(i_1, \dots, i_{j-1}, n)$ containing n of length j. Then

$$(i,n)(i_1, \dots, i_{j-1}, n) = \begin{cases} (i, i_1, \dots, i_{j-1}, n) & i \notin \{i_1, \dots, i_{j-1}\} \\ (i_1, \dots, i_{k-1}, n)(i, i_{k+1}, \dots, i_{j-1}) & i = i_k \end{cases},$$

which is either a $j+1$-cycle containing n or a prodcut of two cycles with total length j. Thus, if ν' is the type of a $\ell(\nu) - 1$-cycle containing n, then

$$X_n t_{\nu'} = c t_\nu + \sum \text{smaller } \ell\text{'s}$$

for some positive integer c. Hence $t_\nu \in \langle Z\mathbf{C}\Sigma_{n-1}, X_n\rangle$. □

1.9. (Oct 03) The spectrum of YJM

Theorem 1.9.0.1. *$GZ(n)$ is generated by $\{X_1, \dots, X_n\}$.*

Proof of Theorem 1.9.0.1. The proof is by induction on n. By the Lemma, $Z\mathbf{C}\Sigma_n \subseteq Z(\Sigma_n, \Sigma_{n-1}) = \langle Z\mathbf{C}\Sigma_{n-1}, X_n\rangle$. By induction,

$$\langle X_1, \dots, X_{n-1}\rangle \supseteq GZ(n-1) \supseteq Z\mathbf{C}\Sigma_{n-1}.$$

Thus

$$\langle X_1, \dots, X_n\rangle \supseteq \langle Z\mathbf{C}\Sigma_{n-1}, X_n\rangle \supseteq Z\mathbf{C}\Sigma_n.$$

We conclude

$$\langle X_1, \dots, X_n\rangle \supseteq \langle GZ(n-1), Z\mathbf{C}\Sigma_n\rangle = GZ(n),$$

as desired. □

Definition 1.9.0.2. $\mathrm{spec}(n) \subseteq \mathbf{C}^n$ is the set of joint eigenvalues of $(X_1, \dots, X_n)$ in all representations of Σ_n. If $\alpha, \beta \in \mathrm{spec}(n)$, the define $\alpha \sim \beta$ if α and β appear as joint eigenvalues in the same representation of Σ_n.

Note that the joint eigenvalues of $Z(\Sigma_n)$ determine the character and thus the irreducible representation of Σ_n. Since $\{X_1, \dots, X_n\}$ generates the Gelfand-Tsetlin algebra $GZ(n) = \langle Z\Sigma_i\rangle \supseteq Z\Sigma_n$, the eigenvalues of $(X_1, \dots, X_n)$ on different representations of Σ_n are disjoint. Thus $\mathrm{spec}(n)/\sim$ is in bijection with the irreducible representations of Σ_n.

1.9.1. Young tableaux.

Definition 1.9.1.1. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \vdash n$, the *Young diagram* is a diagram of left-aligned boxes with λ_i boxes in the ith row.

For example, $\begin{array}{|c|c|c|}\hline &&\\\hline \\\cline{1-1}\end{array}$ corresponds to the partition $(3,1) \vdash 4$.

Definition 1.9.1.2. A *standard Young tableau* is a Young diagram with the numbers $\{1, 2, \ldots n\}$ placed in boxes so that the rows and columns are strictly increasing (here n is the number of boxes).

Note: *tableau* is singular while *tableaux* is plural.

Example 1.9.1.3. The three Young tableaux with shape $(3,1)$ are

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$$

Definition 1.9.1.4. Let T be a standard Young tableau. The *content* of a box $\square$ is

$$c(\square) = x(\square) - y(\square).$$

The *content vector* of T is

$$c(T) = (c(\boxed{1}), \ldots, c(\boxed{n})).$$

Let $\mathrm{Cont}(n)$ be the set of all content vectors of standard Young tableaux of n boxes.

Example 1.9.1.5.

$$c\left(\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right) = (0, 1, -1, 2).$$

Note that the standard Young tableau can be recovered from the content vector.

Define two content vectors $\alpha \approx \beta$ if they correspond to tableaux on the same Young diagram.

Theorem 1.9.1.6 (Branching graph isomorphism). *[Lor18], 4.12 For all n, $\mathrm{Cont}(n) = \mathrm{spec}(n)$ and $\sim=\approx$.*

The theorem characterizes the branching rule for Σ_n. Note that restriction from Σ_n to Σ_{n-1}, at the level of spec, means forgetting the eigenvalues of X_n. Thus, if V^λ is the irreducible representation corresponding to a Young diagram $\lambda \vdash n$, then

$$\mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} V^\lambda = \bigoplus_\mu V^\mu,$$

where $\mu \vdash n-1$ runs over all Young diagrams formed by removing a box from λ.

By deleting the box labelled n from a Young tableau, we find a smaller Young tableau. Thus Young tableau are exactly in bijection with n-tuples

$$\varnothing \subseteq \mu_1 \subseteq \cdots \subseteq \mu_{n-1} \subseteq \lambda$$

where μ_i is a Young diagram with i boxes. Thus the branching graph isomorphism implies that the branching rules for representations of Σ_n and Young diagrams are the same.

First we characterize $\mathrm{Cont}(n)$.

Definition 1.9.1.7. If T is a Young tableau, $s \in \Sigma_n$ is *admissible* if sT is also a Young tableau.

Lemma 1.9.1.8. *Suppose T and T' are λ-tableaux. Then there exists a sequence of admissible adjacent transpositions $s_{i_1}, \ldots, s_{i_\ell}$ such that $T' = s_{i_1} \cdots s_{i_\ell} T'$.*

Proof. It suffices to prove the Lemma when T' is the standard tableau with $1, 2, \ldots, n$ written left to right, top to bottom.

Suppose n is the last entry of the last row of T. Then we can remove that box from both T and T'. By induction, the claim follows.

Now suppose the last entry of the last row of T is n_T. Then $n_T + 1$ is not below or to the right of n_T, so $(n_T, n_T + 1)$ is admissible for T. By induction, there exists a sequence of admissible transpositions taking T to a tableau with n in the last entry of the last row. ☐

Proposition 1.9.1.9. $\mathrm{Cont}(n)$ *is the set of all $\alpha \in \mathbf{C}^n$ such that*

i. $\alpha_1 = 0$;
ii. *at least one of* $\alpha_i \pm 1$ *is in* $\{\alpha_1, \ldots, \alpha_{i-1}\}$ *for all* $i > 1$;
iii. *if* $\alpha_i = \alpha_j$ *for* $i < j$, *then*

$$\{\alpha_i \pm 1\} \subseteq \{\alpha_{i+1}, \ldots, \alpha_{j-1}\}.$$

Proof. First, say that $\alpha \in \mathrm{Cont}(n)$.

i. $\alpha_1 = 0$.
ii. $\alpha_i \pm 1$ is the content of the adjacent boxes to the left and above of i. For λ to be a Young diagram, one of those must also be in the diagram.
iii. Say $\alpha_i = \alpha_j$ for $i < j$. Then all boxes in the rectangle with vertices i and j are in λ. Thus there are boxes between i and j with content $\alpha_i \pm 1$.

Conversely, suppose that α satisfies conditions i-iii. We show $\alpha \in \mathrm{Cont}(n)$ by induction on n. The base case $n = 1$ holds by i. Let $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$. By induction, $\alpha' \in \mathrm{Cont}(n-1)$. Let T' be the Young tableau with content α'. We want to add back in box n to make a tableau T with $c(T) = \alpha$. If the diagonal for α_n is empty, then ii. implies that the nextdoor diagonal is nonempty, so we can add a box to obtain a tableau T. If the diagonal for α_n is nonempty, then there exists $i < n$ such that $\alpha_i = \alpha_n$; let $i < n$ be maximal with this property. Then by iii. there are $r_\pm$ such that $i < r_\pm < n$ and $\alpha_{r_\pm} = \alpha_i \pm 1$. Since T' is standard, $r_\pm$ cannot be above or to the left of i. Thus $r_\pm$ are on the boxes adjacent to i, so we can add n on diagonal α_i to obtain a Young tableau T. ☐

1.10. (Oct 08) Proof of the branching graph isomorphism

1.10.1. Some relations in $\mathbf{C}\Sigma_n$.

Recall the YJM elements

$$X_j = \sum_{i=1}^{j-1} (i, j).$$

Let $s_i = (i, i+1) \in \Sigma_{i+1}$. Then $\{s_1, \ldots, s_{n-1}\}$ generates Σ_n; this is known as the *Coxeter generating set*. We want to understand how s_i acts in the Gelfand-Tsetlin basis. That means we want to understand the operators $s_i X_j$. Note that $s_i X_j = X_j s_i$ if $j \notin \{i, i+1\}$. In the critical case $j \in \{i, i+1\}$,

$$s_i X_i s_i^{-1} = \sum_{j=1}^{i-1} (j, i+1) = X_{i+1} - s_i.$$

This relation may be rewritten as

$$s_iX_i + 1 = X_{i+1}s_i$$

or

$$X_is_i + 1 = s_iX_{i+1}.$$

These relations imply that if v_T is a simultaneous eigenvector for $GZ(n)$, then $\mathbf{C}\{s_iv_T, v_T\}$ is stable under $GZ(n)$, for

$$X_js_iv_T = \begin{cases} s_iX_jv_T & j \notin \{i, i+1\}. \\ X_{i+1}s_iv_T - v_T & j = i \\ X_is_iv_T + v_T & j = i+1 \end{cases}$$

Our calculations above show that if $\{v_T\}$ is a Gelfand-Tsetlin basis vector, then $span\{v_T, s_iV_t\}$ is stable under $\langle X_i, X_{i+1}, s_i\rangle$. Our calculations are based on analyzing this action.

Lemma 1.10.1.1. *Let $\alpha \in \mathrm{spec}(n)$ and let v_α be a GZ basis vector with joint eigenvalue α. Then*

i. *$\alpha_i \neq \alpha_{i+1}$ for all i;*
ii. *if $\alpha_{i+1} = \alpha_i \pm 1$, then $s_iv_\alpha = \pm v_\alpha$;*
iii. *if $\alpha_{i+1} \neq \alpha_i \pm 1$, then*

$$s_i\alpha = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots) \in \mathrm{spec}(n)$$

and $s_i\alpha \sim \alpha$. If $d = (\alpha_{i+1} - \alpha_i)^{-1}$, then

$$v_{s_i\alpha} = s_iv_\alpha - dv_\alpha.$$

Proof. Let $W = span\{v_\alpha, s_iv_\alpha\}$. Then W is stable under $\{s_i, X_i, X_{i+1}\}$.

Suppose $\dim W = 1$. Then $s_iv_\alpha = \pm v_\alpha$ and

$$X_{i+1}v_\alpha = s_iX_is_iv_\alpha + s_iv_\alpha = (\alpha_i \pm 1)v_\alpha,$$

so $\alpha_{i+1} = \alpha_i \pm 1$ and $s_iv_\alpha = \pm v_\alpha$ in this case.

Now suppose $\dim W = 2$. Then in the basis v_α, s_iv_α, the operators X_i and X_{i+1} have matrices

$$X_i \mapsto \begin{bmatrix} \alpha_i & -1 \\ 0 & \alpha_{i+1} \end{bmatrix} \qquad X_{i+1} \mapsto \begin{bmatrix} \alpha_{i+1} & 1 \\ 0 & \alpha_i \end{bmatrix}$$

Since X_i and X_{i+1} are diagonalizable on $V(\alpha)$, they are also diagonalizable on W, so $\alpha_i \neq \alpha_{i+1}$. Thus we have shown $\alpha_i \neq \alpha_{i+1}$ in all cases. If we set $d = (\alpha_{i+1} - \alpha_i)^{-1}$ and $w = s_iv_\alpha - dv_\alpha$, then

$$X_iw = \alpha_{i+1}s_iv_\alpha - v_\alpha - \alpha_idv_\alpha = \alpha_{i+1}(s_iv_\alpha - dv_\alpha).$$

Similarly, $X_{i+1}w = \alpha_iw$. Thus w is a Gelfand-Tsetlin basis vector for $s_i\alpha$. The matrix for s_i in the basis $\{v_\alpha, w\}$ for W is given by

$$s_iv_\alpha = w + dv_\alpha$$
$$s_iw = v_\alpha - ds_iv_\alpha = (1 - d^2)v_\alpha - dw.$$
$$s_i \mapsto \begin{bmatrix} d & 1 - d^2 \\ 1 & -d \end{bmatrix}.$$

If $d = \pm 1$, then the matrix for s_i is $\begin{pmatrix} \pm 1 & 0 \\ 1 & \mp 1 \end{pmatrix}$. If $\langle , \rangle$ is an invariant Hermitian form on our representation, then $\langle s_ix, s_iy\rangle = \langle x, y\rangle$ for all x, y (by definition of invariance). Lemma 1.8.1.10 says that the Gelfand-Tsetlin basis is orthogonal with respect to any invariant Hermitian form. But when $d = \pm 1$,

$$\begin{aligned} 0 = \langle v_\alpha, w\rangle &= \langle s_i v_\alpha, s_i w\rangle \\ &= \langle \pm v_\alpha + w, \mp w\rangle \\ &= \mp\langle w, w\rangle \neq 0, \end{aligned}$$

a contradiction. Thus $d \neq \pm 1$. The proof is complete. □

Recall Coxeter relations $s_i s_j = s_j s_i$ for $|i - j| > 1$ and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

Lemma 1.10.1.2. *i.* $\operatorname{spec}(n) \subseteq \operatorname{Cont}(n)$

ii. if $\alpha \in \operatorname{spec}(n), \beta \in \operatorname{Cont}(n)$, $\alpha \approx \beta$, *then* $\beta \in \operatorname{spec}(n)$ *and* $\beta \sim \alpha$, *as desired.*

Proof. i. Say $\alpha \in \operatorname{spec}(n)$. Since $X_1 = 0$, $\alpha_1 = 0$.

Suppose towards contradiction that neither of $\alpha_i \pm 1$ is in $\{\alpha_1, \ldots, \alpha_{i-1}\}$ for some $i > 1$. Let i be least with this property. Then by Lemma 1.10.1.1, $(\alpha_1, \ldots, \alpha_{i-2}, \alpha_i, \alpha_{i-1}, \alpha_{i+1}, \ldots) \in \operatorname{spec}(n)$. By induction we conclude

$$(\alpha_i, \alpha_1, \ldots, \hat{\alpha_i}, \ldots) \in \operatorname{spec}(n).$$

Thus $\alpha_i = 0$, which contradicts Lemma 1.10.1.1 since adjacent spectral values must be distinct.

Now say $i < j$ such that $\alpha_i = \alpha_j$ and $\{\alpha_i \pm 1\} \not\subseteq \{\alpha_{i+1}, \ldots, \alpha_{j-1}\}$. Pick $j - i$ to be minimal If $\alpha_{i+1} \neq \alpha_i \pm 1$ or $\alpha_j \neq \alpha_i \pm 1$, then we can swap, contradicting minimality. If $j - i = 1$, then we have $(\ldots, \alpha_i, \alpha_i, \ldots)$, contradicting Lemma 1.10.1.1. Thus α is of the form

$$(\ldots, a, a \pm 1, \ldots, a \pm 1, a, \ldots).$$

By minimality, a does not appear between $a \pm 1$ and $a \pm 1$. But then $\ldots, a \pm 1, \ldots, a \pm 1, \ldots$ is a smaller examle of what we seek, unless both $a \pm 1$ are in the same place in the vector. Thus, the minimal $\alpha \in \operatorname{spec}(n)$ such that $\alpha_i = \alpha_j$ and $\{\alpha_i \pm 1\} \not\subseteq \{\alpha_{i+1}, \ldots, \alpha_{j-1}\}$ must be of the form

$$\alpha = (\ldots, a, a \pm 1, a, \ldots),$$

i.e. $\alpha_{i+1} = \alpha_i \pm 1$ and $\alpha_{i+2} = \alpha_{i+1} \mp 1$. By Lemma 1.10.1.1, $s_i v_\alpha = \pm v_\alpha$ and $s_{i+1} v_\alpha = \mp v_\alpha$. Thus

$$\pm v_\alpha = s_{i+1} s_i s_{i+1} v_\alpha = s_i s_{i+1} s_i v_\alpha = \mp v_\alpha,$$

a contradiction! We conclude α satisfies property iii of content vectors.

Thus $\alpha \in \operatorname{Cont}(n)$, as desired.

ii. Suppose $\alpha \in \operatorname{spec}(n)$, $\beta \in \operatorname{Cont}(n)$, and α and β correspond to the same Young diagram. By Lemma 1.9.1.8, there exists a sequence $s_{i_1}, \ldots, s_{i_k}$ of admissible transpositions such that

$$T_\beta = s_{i_1} \cdots s_{i_k} T_\alpha.$$

Admissibility means the boxes $i, i + 1$ we want to swap are not adjacent, so $c_i \neq c_{i+1} \pm 1$. Thus by Lemma 1.10.1.1,

$$s_{i_j} \cdots s_{i_k} v_\alpha \in \mathbf{C}^* v_{s_{i_j} \cdots s_{i_k}} v_\alpha,$$

so

$$\alpha \sim s_{i_j} \cdots s_{i_k} \alpha \in \mathrm{spec}(n).$$

Thus $\beta \in \mathrm{spec}(n)$ and $\alpha \sim \beta$, as desired.

□

Theorem 1.10.1.3. *For all n, $\mathrm{spec}(n) = \mathrm{Cont}(n)$ and $\sim = \approx$. The branching graph for the symmetric groups is the branching graph for Young diagrams.*

Proof of Theorem 1.10.1.3. By Lemma 1.10.1.2, $\mathrm{spec}(n) \subseteq \mathrm{Cont}(n)$, and for each equivalence class $[c] \in \mathrm{Cont}(n)/\approx$, $[c] \cap \mathrm{spec}(n)$ is contained in a single equivalence class. (As Jameson points out, we have shown that equivalence classes of $\mathrm{spec}(n)$ are unions of equivalence classes in $\mathrm{Cont}(n)$). Thus $|\,\mathrm{spec}(n)/\sim| \leq |\,\mathrm{Cont}(n)/\approx|$. But both sets have size equal to the number of partitions of n, which means that each equivalence class in $\mathrm{Cont}(n)$ is an equivalence class in $\mathrm{spec}(n)$, and $\mathrm{spec}(n) = \mathrm{Cont}(n)$. □

1.11. (Oct 10) Murnaghan-Nakayama rule

1.11.1. Rimhooks. Statement of the rule. The Murnaghan-Nakayama rule gives a combinatorial description of the character values of Σ_n.

Definition 1.11.1.1. For a Young diagram λ, the *boundary* consists of those boxes with no southwest neighbor. A connected subset of the boundary is called a *rimhook*. If a rimhook has m boxes, we will call it an m-rimhook. The *height $ht(\nu)$* of a rimhook ν is the number of rows in the rimhook plus one.

Theorem 1.11.1.2 (Murnaghan-Nakayama rule). *Let $\lambda \vdash n$. Suppose that $\sigma \in \Sigma_n$, and that $\sigma = c\sigma'$ for c a cycle of length h and σ' a permutation disjoint from c. Then*

$$\chi^\lambda(\sigma) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu \text{ } h\text{-rimhook}}} \chi^\mu(\sigma')(-1)^{\mathrm{ht}(\lambda/\mu)}$$

We first prove the Murnaghan-Nakayama rule for an n-cycle (when $\mu = \varnothing$).

Lemma 1.11.1.3. *Let $c \in \Sigma_n$ be an n-cycle. Then*

$$\chi^\lambda(c) = \begin{cases} (-1)^h & \lambda = (n-h, 1^h) \text{ is a hook} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note $X_2 \cdots X_n$ is the sum of all n-cycles in Σ_n. Thus

$$\chi^\lambda(c) = \frac{1}{(n-1)!} \mathrm{tr}(X_2 \cdots X_n; V^\lambda).$$

If λ is not a hook, then every content vector has a 0 after the 1st position, so $X_2 \cdots X_n V^\lambda = 0$.

Suppose λ is a hook. Note that a tableau is determined exactly by which of the numbers $\{2, 3, \ldots, n\}$ go into the "leg" of h boxes, so the total number is $\binom{n-1}{h}$. On the standard tableau, the content is

$$(0, 1, \ldots, n-h-1, -1, \ldots, -h).$$

Thus $X_2 \cdots X_n$ has eigenvalue $(n-h-1)!h!(-1)^h$. But $X_2 \cdots X_n \in \mathbf{ZC}\Sigma_n$ since it is the sum of all n-cycles, so it acts by the scalar $(n-h-1)!h!(-1)^h$. Thus

$$\frac{1}{(n-1)!}\operatorname{tr}(X_2 \cdots X_n; V^\lambda) = \frac{1}{(n-1)!}\binom{n-1}{h}(n-h-1)!h!(-1)^h = (-1)^h,$$

as desired. □

1.11.2. Skew diagrams and skew tableaux. Let $\Sigma'_{n-k} \subseteq \Sigma_n$ be the permutations stabilizing $\{1, \ldots, k\}$ elementwise. Then $\Sigma'_{n-k} \subseteq Z(\Sigma_n, \Sigma_k)$,

Definition 1.11.2.1. If $\mu \subseteq \lambda$ are Young diagrams of size k and n, then

$$V^{\lambda/\mu} = \operatorname{Hom}_{\mathbf{C}\Sigma_k}(V_\mu, \operatorname{Res}_{\Sigma_k} V_\lambda),$$

a module over $\Sigma'_{n-k} \subset Z(\Sigma_n, \Sigma_k)$.

By the branching rule, a basis for $V^{\lambda/\mu}$ is indexed by chains $\mu = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \lambda_{n-k} = \lambda$. These correspond to *skew tableaux*: filling the skew shape λ/μ with numbers $\{1, \ldots, n-k\}$ such that rows are increasing and columns are increasing downwards. The number of skew tableaux depends only on the shape of λ/μ and not the particular choice of λ or μ with this difference.

Theorem 1.11.2.2 ([Lor18], Theorem 4.22). *The skew hook module $V^{\lambda/\mu}$ depends only on the shape of λ/μ.*

Lorenz's proof of this theorem involves working with an explicit basis. It would be nice to know a basis-free proof!

The MN rule is equivalent to:

Theorem 1.11.2.3. *Let s be a cycle of length $n-k$. Then*

$$\operatorname{tr}(s; V^{\lambda/\mu}) = \begin{cases} (-1)^{ht(\lambda/\mu)} & \lambda/\mu \text{ is a rim hook} \\ 0 & \lambda/\mu \text{ is not a rim hook} \end{cases}$$

If $\gamma \vdash n$, let $\Sigma_\gamma = \Sigma_{\gamma_1} \times \Sigma_{\gamma_2} \times \cdots \subseteq \Sigma_n$.

Lemma 1.11.2.4 (Restriction to Young subgroups. [Lor18], 4.27, Step 1). *Let $\lambda \vdash n$ and let $\mu \vdash k, \mu \subseteq \lambda$. Let $\gamma \vdash n-k$. Then*

$$\operatorname{Res}^{\Sigma'_{n-k}}_{\Sigma'_\gamma} V^{\lambda/\mu} = \bigoplus_{\Lambda} V^{\lambda_1/\lambda_0} \boxtimes \cdots \boxtimes V^{\lambda_\ell/\lambda_{\ell-1}}$$

where Λ is the set of chains

$$\mu = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_\ell = \lambda$$

where $|\lambda_i/\lambda_{i-1}| = \gamma_i$. The induced maps

$$\operatorname{Ind}^{\Sigma'_{n-k}}_{\Sigma'_\gamma} V^{\lambda_1/\lambda_0} \boxtimes \cdots \boxtimes V^{\lambda_\ell/\lambda_{\ell-1}} \to V^{\lambda/\mu}$$

are surjective.

Lemma 1.11.2.5 ([Lor18],Lemma 4.27, Step 2). *Suppose λ/μ is disconnected and s is an $n-k$-cycle. Then*

$$\operatorname{tr}(s; V^{\lambda/\mu}) = 0.$$

Proof. Write $\lambda/\mu = \lambda_1/\mu \sqcup \lambda_2/\mu$ and let $\gamma = (|\lambda_1/\mu|, |\lambda_2/\mu|)$. Then we obtain a surjective map

$$\operatorname{Ind}_{\Sigma'_\gamma}^{\Sigma_{n-k}} V^{\lambda_1/\mu} \boxtimes V^{\lambda_2/\mu} \to V^{\lambda/\mu}$$

By a dimension count, this map is an isomorphism.

Thus $V^{\lambda\mu}$ is induced from the subgroup Σ_γ. As s is not conjugate into Σ_γ, we conclude $tr(s; V^{\lambda/\mu}) = 0$. ☐

Lemma 1.11.2.6 ([Lor18], Proposition 4.28). *Suppose λ/μ is not contained in the boundary. Then*

$$\operatorname{tr}(s; V^{\lambda/\mu}) = 0.$$

Proof. Since λ/μ is not contained in the boundary, λ/μ contains the Young diagram for $(2,2)$. Let $\gamma = (4, 1^{n-k-4})$; then $\Sigma'_\gamma \cong \Sigma_4$. We find an epimorphism

$$\operatorname{Ind}_{\Sigma_4}^{\Sigma'_{n-k}} V^{(2,2)} \to V^{\lambda/\mu}.$$

Now if V^α is an irreducible constituent of $V^{\lambda/\mu}$, it appears in $\operatorname{Ind}_{\Sigma_4} V^{(2,2)}$. By the branching rule, this implies $(2,2) \subseteq \alpha$, so α is not a hook. Thus $\operatorname{tr}(s; V^\alpha) = 0$. As this holds for all constituents V^α, we conclude

$$\operatorname{tr}(s; V^{\lambda/\mu}) = 0.$$ ☐

Lemma 1.11.2.7 ([Lor18], Proposition 4.28). *Suppose λ/μ is a rimhook of size $h = n - k$. Let $\nu = (n-k-h, 1^h)$ be a hook. Then*

$$[V^\nu : V^{\lambda/\mu}] = \begin{cases} 1 & h = ht(\lambda/\mu) \\ 0 & else \end{cases}$$

Proof. Since the representation $V^{\lambda/\mu}$ depends only on the shape of λ/μ, we may assume λ/μ touches the axes $y = 0$ and $x = 0$. A hook $\nu = (n-k-h, 1^h)$ is contained in λ only if $h = ht(\lambda/\mu)$. Since $V^{\lambda/\mu} \subseteq \operatorname{Res}_{\Sigma'_{n-k}}^{\Sigma_n} V^\lambda$, by the branching rule we obtain $[V^\nu : V^{\lambda/\mu}] = 0$ in this case.

Now let ν be the hook with $h = ht(\lambda/\mu)$. We have

$$V^\lambda|_{\Sigma_k \times \Sigma_{n-k}} = \bigoplus_\alpha V^\alpha \boxtimes V^{\lambda/\alpha}.$$

But by the same token

$$V^\lambda|_{\Sigma_k \times \Sigma_{n-k}} = \bigoplus_\beta V^{\lambda/\beta} \boxtimes V^\beta.$$

But $\lambda/\nu \cong \mu/\varnothing = \mu$ as representations of Σ_k, as λ/μ depends only on the shape.

The first decopmosition gives

$$[V^\mu \boxtimes V^\nu : \operatorname{Res}_{\Sigma_k \times \Sigma_{n-k}} V^\lambda] = [V^\nu : V^{\lambda/\mu}],$$

while the second gives

$$[V^\mu \boxtimes V^\nu : \operatorname{Res}_{\Sigma_k \times \Sigma_{n-k}} V^\lambda] = [V^\mu : V^{\lambda/\nu}] = 1,$$

as desired. ☐

CHAPTER 2

Representations of the unitary group

2.1. (Oct 17) Topological groups and compact groups

2.1.1. Topological groups.

Definition 2.1.1.1. A *topological group* G is a topological space G, equipped with continuous maps $m : G \times G \to G$, $i : G \to G$, and $e : * \to G$ which make G into a group with product m and inverse $g^{-1} = i(g)$, and such that the topology on G is Hausdorff.

A topological group G is *compact* is G is compact as a topological space.

Example 2.1.1.2. If G is a group in the ordinary sense, then G is a topological group under the discrete topology. A discrete group is compact if and only if it is finite.

Example 2.1.1.3. G is a *Lie group* if G is a smooth manifold and the multiplication and inverse maps are morphisms of manifolds. A Lie group is a topological group.

For example, $G = GL_n(\mathbf{R})$ is a Lie group since G is an open subset of $M_{n\times n}(\mathbf{R})$ and the multiplication and inverse maps are rational functions.

Example 2.1.1.4. The unitary group is

$$U(n) = \{x \in GL_n(\mathbf{C}) \mid x^H x = 1\},$$

where x^H is the Hermitian conjugate of x: $x^H = \bar{x}^t$. Note that $U(n)$ is a closed subgroup of $GL_n(\mathbf{C})$. It is also a Lie group. Since $|x_{ij}| \leq 1$ for all $x \in U(n)$, $U(n)$ is a closed subspace of $\bar{\mathbb{D}}^{n^2}$ where $\bar{\mathbb{D}} \subseteq \mathbf{C}$ is the closed unit disk. Thus $U(n)$ is compact.

Example 2.1.1.5. Suppose $\cdots \to G_2 \to G_1$ is an inverse system of finite groups, with transition maps $t_i : G_i \to G_{i-1}$. Then $G = \varprojlim_i G_i$ carries the *inverse limit* topology. We can realize $G \subseteq \prod_i G_i$ as the subset

$$G = \left\{ (g_i)_{i\geq 1} \in \prod_i G_i \,\middle|\, t_i(g_i) = g_{i-1} \text{ for all } i \right\}.$$

The inverse limit topology is the subspace topology of the product topology on $\prod_i G_i$, where each G_i has the discrete topology. There are continuous maps $\pi_i : G \to G_i$. A basis for the topology is given by the sets $\pi_i^{-1}(x)$ as $i \geq 1$ and $x \in G_i$.

By Tychonoff's theroem, $\prod_i G_i$ is compact, so G, being a closed subspace of a compact space, is also compact.

An example of a profinite group is $GL_n(\mathbf{Z}_p)$, where $\mathbf{Z}_p$ is the ring of p-adic integers.

Example 2.1.1.6. $G = GL_n(\mathbf{Q}_p)$, with the metric topology induced from $\mathbf{Q}_p$, is a topological group. It is the union of conjugates of the profinite group $GL_n(\mathbf{Z}_p)$.

2.1.2. Haar integral.

Recall that if X is a topological space, we let $C(X, \mathbf{C})$ denote the space of continuous functions $X \to \mathbf{C}$.

Theorem 2.1.2.1 ([Wei40], Chapter II). *Let G be a compact group. Then there exists a unique linear function*

$$\int_G : C(X, \mathbf{C}) \to \mathbf{C},$$
$$f \mapsto \int_G f = \int_G f(g)dg$$

called the Haar integral*, such that*

i. if $f \geq 0$*, then* $\int_G f \geq 0$*;*

ii. $\int_G$ *satisfies the* triangle inequality:

$$\left| \int_G f \right| \leq \int_G |f|$$

iii. $\int_G 1 = 1$*.*

iv. $\int_G$ *is* biinvariant*:*

$$\int_G f(xg)dg = \int_G f(g)dg = \int_G f(gx)dg$$

for all $x \in G$*.*

The first three conditions imply that $\int_G : C(X, \mathbf{C}) \to \mathbf{C}$ is *continuous*. The Riesz-Markov theorem says that continuous, positive linear functionals on $C(X, \mathbf{C})$ are given by integration with respect to a Borel measure. The measure associated to the Haar integral is called the *Haar measure*.

Remark 2.1.2.2. If an integral on a compact group G satisfies i.-iii. and is left invariant, then it is also right invariant.

We will not prove the theorem in general. In the cases we care about, it is easier to construct the Haar integral than to prove the theorem in general.

Example 2.1.2.3. Let G be a finite group. Then

$$\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Example 2.1.2.4. Let G be a compact Lie group. Since G is a manifold, we may take a volume form $\omega_e \in \wedge^\top T_e^* G$. Define a differential form ω by $\omega_g = \ell_{g^{-1}}^* \omega_e$, where $\ell_h : G \to G$ is left translation by h. Then ω is a left-invariant volume form, so by the theory of integration on manifolds,

$$\int_G f = \int_G f\omega$$

satisfies i.-iii. and is left invariant.

Example 2.1.2.5. Let G be a profinite group. The Haar integral will be uniquely specified by the integral of the characteristic functions 1_X where X is an open set. If $K_n = \ker \pi_n : G \to G_n$, then

$$\int_G 1_{K_n} = \frac{1}{[G : K_n]} = \frac{1}{|G_n|}.$$

2.1.3. Integrating vector-valued functions (Calc III).

Let V be a finite-dimensional vector space. Then $\int_G$ defines an integral

$$\int_G : C(G, V) \to V$$

as follows: given a basis for V, we get an isomorphism $V \cong \mathbf{C}^n$, and then

$$\int_G : C(G, V) \cong C(G, \mathbf{C}^n) \to \mathbf{C}^n \cong V$$

is defined to be integration component-wise. Linearity of the one-dimensional integral implies that the resulting function does not depend on the choice of basis.

2.1.4. Generalities on representations of compact groups.

Definition 2.1.4.1. A *representation* of a topological group G is a continuous homomorphism

$$\rho : G \to GL(V)$$

into a *finite-dimensional* complex vector space V.

There are also infinite-dimensional representations, but we will not consider them in this course. A morphism of representations $V \to W$ is a linear map $T : V \to W$ such that $gT = Tg$ for all $g \in G$.

Theorem 2.1.4.2. *Let V be a representation of a compact group G. If $U \subseteq V$ is a subrepresentation, then there exists a subrepresentation $W \subseteq V$ such that $V = U \oplus W$.*

Proof. Again there exists a linear map $\tilde{\pi} : V \to U$ such that $\pi|_U = \mathrm{id}_U$. Define

$$\pi = \int_G g\tilde{\pi}g^{-1}dg \in \mathrm{Hom}(V, U).$$

Then π is G-linear and $\pi|_U = \int_G g\mathrm{id}_U g^{-1}dg = \mathrm{id}_U$. Then $W = \ker \pi$ is a subrepresentation of V and satisfies $V = W \oplus U$. □

An *irreducible* representation V has no proper nonzero subrepresentation. We have the character $\chi_V : G \to \mathbf{C}$ of a representation. It is continuous.

Theorem 2.1.4.3. *Let V, V' be representations of compact G. Then*

$$\dim \mathrm{Hom}_G(V, V') = \int_G \chi_V(g^{-1})\chi_{V'}(g)dg.$$

For a representation V, the matrix entries define a function

$$act_V^* : V^* \otimes V \to C(G, \mathbf{C}).$$

Note act_V^* is a $G \times G$-linear map.

Theorem 2.1.4.4 (Peter-Weyl. [Ada69], Theorem 3.39).

$$L^2(G) = \widehat{\bigoplus_{V \text{ simple } G\text{-rep}}} V^* \otimes V$$

over $G \times G$.

Again, we will not prove it.

Example 2.1.4.5. If $G = S^1 \cong \mathbf{R}/\mathbf{Z}$, then for each $n \in \mathbf{Z}$ we have $\rho_n : S^1 \to \mathbf{C}^\times$ given by $\rho_n(x) = \exp(2\pi i n x)$. Then Fourier analysis tells us that $L^2(S^1) = \widehat{\bigoplus_{n\in\mathbf{Z}}\mathbf{C}} \cdot \rho_n$.

2.1.5. Matrix groups. [GW09], §1.3.1-2.

Definition 2.1.5.1. A *linear group* is a closed subgroup of some $GL_n(\mathbf{R})$.

When working with linear groups, we can work directly with analysis in $M_n(\mathbf{R})$, the space of $n \times n$ matrices.

Definition 2.1.5.2. Fix a norm $|\cdot|$ on $\mathbf{R}^n$. The *matrix norm* on $A \in M_n(\mathbf{R})$ is defined by

$$|A| = \sup_{|x|=1} |Ax|.$$

The supremum exists and is attained since the unit sphere is compact. By definition, if $x \neq 0$,

$$|Ax| = \left| A \frac{x}{|x|} \right| |x| \leq |A||x|.$$

The matrix norm is *submultiplicative:*

$$|AB| = \sup_{|x|=1} |ABx| \leq |A| \sup_{|x|=1} |Bx| \leq |A||B|.$$

Definition 2.1.5.3. The *matrix exponential* of $a \in M_n(\mathbf{R})$ is

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}.$$

Since the norm $|\cdot|$ is submultiplicative, we have

$$\left| \sum_{n=k}^{\ell} \frac{a^n}{n!} \right| \leq \sum_{n=k}^{\ell} \frac{|a^n|}{n!}.$$

Hence the series for e^a converges absolutely and thus converges for all $a \in M_n(\mathbf{R})$. Furthermore, $a \mapsto e^a$ is continuous.

2.2. (Oct 22) The Lie algebra of a linear group

2.2.1. Matrix identities.

Theorem 2.2.1.1. *Let $f \in \mathbf{C}[\![t_1, \ldots, t_n]\!]$ be a power series converging on the polydisk $\{t \in \mathbf{C}^n \mid |t_i| < R\}$ and let $A_1, \ldots, A_n$ be pairwise commuting matrices. If $|A_i| < R$ for all i, then $f(A_1, \ldots, A_n)$ converges absolutely. The assignment*

$$(A_1, \ldots, A_n) \to f(A_1, \ldots, A_n)$$

is a smooth function on its domain, and is compatible with addition, multiplication, and composition of formal power series.

EXAMPLE 2.2.1.2. Fix $A \in M_n(\mathbf{R})$. Then $t \mapsto e^{tA}$ is a smooth function $\mathbf{R} \to GL_n(\mathbf{R})$. If we view $GL_n(\mathbf{R}) \subseteq M_n(\mathbf{R})$, the derivative of this function is $t \mapsto Ae^{tA}$.

EXAMPLE 2.2.1.3. Recall that $e^{x+y} = e^x e^y$ as power series in commuting variables x and y. By compatibility with multiplication, $e^{A+B} = e^A e^B$ if A and B commute.

EXAMPLE 2.2.1.4. Fix a matrix A. As sA and tA commute for $s, t \in \mathbf{R}$, we see $t \mapsto e^{tA}$ is a homomorphism $\mathbf{R} \to GL$, i.e. $e^{(s+t)A} = e^{sA}e^{tA}$.

EXAMPLE 2.2.1.5. The Taylor series for

$$\log(1+x) = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$$

converges for $|x| < 1$. Thus if $|A - I_n| < 1$, then the series

$$\log(A) = \sum_{n \geq 1} (-1)^{n+1} \frac{(A-I)^n}{n}$$

converges. In this domain, log is a smooth inverse for exp.

2.2.2. The Lie algebra of a linear group. [GW09], §1.3.3.

DEFINITION 2.2.2.1. A *linear group* is a closed subgroup $G \subseteq GL_n(\mathbf{R})$.

Note that a linear group comes with an embedding into a fixed $GL_n(\mathbf{R})$.

EXAMPLE 2.2.2.2. $U(n) = \{x \in GL_n(\mathbf{C}) \mid x\bar{x}^T = 1\}$ is a linear group (Homework).

DEFINITION 2.2.2.3. The *Lie algebra* of a linear group $G \subseteq GL_n(\mathbf{R})$ is

$$L(G) = \{a \in M_n(\mathbf{R}) \mid e^{ta} \in G \forall t \in \mathbf{R}\}$$

LEMMA 2.2.2.4 (Lim-lemma). *Suppose $G \subseteq GL_n(\mathbf{R})$ is a linear group. Let $\{x_i\}$ be a sequence in $M_n(\mathbf{R})$ such that $x_i \neq 0$ for all i, $e^{x_i} \in G$ for all i, $x_i \to 0$, and $x_i/|x_i| \to x$ for some $x \in M_n(\mathbf{R})$. Then $e^{tx} \in G$ for all $t \in \mathbf{R}$.*

PROOF. Let $t \in \mathbf{R}$. For each i, subdivide $\mathbf{R}$ into intervals of size $|x_i|$. Thus there exists $m_i \in \mathbf{Z}$ such that $|t - m_i|x_i|| \leq |x_i|$. Since $x_i \to 0$, $\lim_{i \to \infty} m_i|x_i| = t$. Then

$$\lim_{i \to \infty} m_i x_i = \lim_{i \to \infty} (m_i|x_i|) \frac{x_i}{|x_i|} = tx.$$

By continuity of the exponential function,

$$e^{tx} = \lim_{i \to \infty} e^{m_i x_i} = \lim_{i \to \infty} (e^{x_i})^{m_i}.$$

As $e^{x_i} \in G$ and $m_i \in \mathbf{Z}$, we see $(e^{x_i})^{m_i} \in G$. Since G is closed, $e^{tx} \in G$. ☐

LEMMA 2.2.2.5 ([GW09], Lemma 1.3.6). *Suppose $x, y \in M_n(\mathbf{R})$. Then for N sufficiently large,*

$$\log(e^{x/N}e^{y/N}) = \frac{X+Y}{N} + \frac{[X,Y]}{2N^2} + \frac{\epsilon_N}{N^2}$$

where $\epsilon_N \to 0$.

Proof. Expand $\log(e^{sX}e^{tY})$ into power series and collect all terms with degrees > 2 into β. At the end we'll take $s = t = 1/N$.

Now

$$\begin{aligned}\log(e^{sX}e^{tY}) &= (e^{sX}e^{tY} - 1) - \frac{1}{2}\left(e^{sX}e^{tY} - 1\right)^2 + \cdots \\ &= \left(sX + tY + s^2X^2/2 + stXY + t^2Y^2/2\right) - \frac{1}{2}(s^2X^2 + st(XY + YX) + t^2Y^2) + \cdots \\ &= sX + tY + \frac{1}{2}st(XY - YX) + \cdots . \end{aligned}$$

☐

Corollary 2.2.2.6. *Suppose* $x, y \in M_n(\mathbf{R})$. *Then*

i.

$$e^{x/N}e^{y/N} = e^{1/N(x+y+\alpha_N)}$$

where $\alpha_N \to 0$ *as* $N \to \infty$.

ii.

$$e^{x/N}e^{y/N}e^{-(x+y)/N} = e^{1/2N^2([X,Y]+\beta_N)}$$

where $\beta_N \to 0$ *as* $N \to \infty$.

Proof. For part ii, apply Lemma 2.2.2.5 twice. Set $Z_N = N\log(e^{x/N}e^{y/N})$ for N sufficiently large. Then

$$\begin{aligned}\log(e^{x/N}e^{y/N}e^{-(x+y)/N}) &= \log(e^{Z_N/N}e^{-(x+y)/N}) \\ &= \frac{Z_N - (x+y)}{N} + \frac{1}{2}[Z_N, -(x+y)]N^2 + o(1/N^2).\end{aligned}$$

But $Z_N/N = \frac{x+y}{N} + \frac{[x,y]}{2N^2} + o(1/N^2)$, so $[Z_N, -(x+y)]/N^2 = o(1/N^2)$, and

$$\log(e^{x/N}e^{y/N}e^{-(x+y)/N}) = \frac{[x,y]}{2N^2} + o(1/N^2),$$

as desired. ☐

Proposition 2.2.2.7. *Let* $G \subseteq GL_n(\mathbf{R})$ *be a linear group.*

i. $L(G)$ *is a subspace of* $M_n(\mathbf{R})$*;*
ii. $L(G)$ *is closed under the commutator* $[X, Y] = XY - YX$.

Proof. First, note that $L(G)$ is closed under scaling by definition. If $x, y \in L(G)$, then $e^{x/N}e^{y/N} \in G$ for all $N \geq 1$. Assume $x \neq -y$. Apply the Lemma to

$$x_N = \log(e^{x/N}e^{y/N}) = \frac{1}{N}(x + y + \alpha_N),$$

where $\alpha_N \to 0$. Then

$$\lim_{N\to\infty} \frac{x_N}{|x_N|} = \frac{x+y}{|x+y|}$$

so $x + y \in L(G)$ by the Lim lemma.

Simiarly, if $x, y \in L(G)$, then $e^{x/N}e^{y/N}e^{-(x+y)/N} \in G$ for all $N \geq 1$. Applying the Lim Lemma to

$$x_N = \log(e^{x/N}e^{y/N}e^{-(x+y)/N}) = \frac{1}{2N^2}([x,y] + \beta_N)$$

where $\beta_N \to 0$ gives $[x, y] \in L(G)$. ☐

THEOREM 2.2.2.8 (E. Cartan). *Let G be a linear group. Then there is a neighborhood $0 \in U$ in $M_n(\mathbf{R})$ and a C^∞ diffeomorphism $f : U \to f(U) \ni 1$ such that $f(U) \subseteq GL_n(\mathbf{R})$ is open and $f : U \cap L(G) \cong f(U) \cap G$.*

Slogan: "every linear group is a Lie group"

PROOF. Note that if $G = GL_n$, exp does the job. In general, let $V \subseteq M_n(\mathbf{R})$ be a complementary subspace to $L(G)$, and define $f : M_n(\mathbf{R}) = L(G) \oplus V \to GL_n(\mathbf{R})$ by

$$f(u \oplus v) = \exp(u)\exp(v).$$

Note that $df_0 = id$. By the inverse function theorem, there is a neighborhood $U \ni 0$ such that $f|_U$ is a diffeomorphism $U \to f(U)$. By shrinking, we may assume $U = U' \times U''$ for $U' \subseteq L(G)$ and $U'' \subseteq V$. Then

$$f(U' \times U'') = \exp(U')\exp(U'').$$

Suppose towards contradiction that every neighborhood $U = U' \times U'' \ni 0$ has $f(U \cap L(G)) \neq f(U) \cap G$. As $f(U' \times 0) \subseteq G$ by definition of Lie algebra, there exists $v'_n \oplus v''_n \to 0$, $v''_n \neq 0$ such that $f(v'_n + v''_n) \in G$. Then $f(v'_n) \in G$, so $f(v''_n) \in G$. Since the unit ball in V is compact, $v''_n/\|v''_n\|$ has a convergent subsequence. Replace v''_n with this subsequence. Then

$$x = \lim_{n\to\infty} \frac{v''_n}{\|v''_n\|} \in V$$

Note $\|x\| = 1$, so $x \neq 0$. By the Lim-lemma, $e^{tx} \in G$ for all t, i.e. $x \in L(G)$. Thus $x \in L(G) \cap V = 0$, so $x = 0$, a contradiction.

Thus there is a neighborhood $U = U' \times U'' \ni 0$ such that $f(U) \cap G = f(U \cap L(G))$. □

In the proof, we see that $f|_{L(G)} = \exp$, and so there is a neighborhood $0 \in U' \subseteq L(G)$ such that $\exp(U') \subseteq G$ is an open neighborhood of $1 \in G$.

2.3. (Oct 24) Representations of linear groups

2.3.1. The differential of a homomorphism. Let $\rho : G \to GL(V)$ be a representation of a linear group G (i.e. V is a complex finite-dimensional vector space and ρ is a continuous homomorphism).

LEMMA 2.3.1.1. *Let $f : \mathbf{R} \to GL(V)$ be a continuous homomorphism. Then there is a unique $A \in M_n(\mathbf{C})$ such that $f(t) = e^{tA}$ for all $t \in \mathbf{R}$.*

PROOF. Recall the exponential map $\exp : M_n(\mathbf{C}) \to GL_n(\mathbf{C})$ has a neighborhood $0 \in U \subseteq M_n(\mathbf{C})$ such that $\exp : U \cong \exp(U) = V$, with inverse $\log : V \to U$. Recall that $e^{X+Y} = e^X e^Y$ if X and Y commute; thus $\log(AB) = \log(A) + \log(B)$ when all three make sense. Let $\epsilon > 0$ be such that $\log(f(t))$ is defined for $|t| < \epsilon$. Thus $\log(f(t+s)) = \log(f(t)f(s)) = \log(f(t)) + \log(f(s))$ whenever $s, t, s+t \in (-\epsilon, \epsilon)$. Thus $\log f$ is an additive continuous function $(-\epsilon, \epsilon) \to M_n(\mathbf{C})$, so there is a unique $A \in M_n(\mathbf{C})$ such that $\log f(t) = tA$ for $|t| < \epsilon$. Hence $f(t) = e^{tA}$ for sufficiently small t. For general t, pick $n \in \mathbf{Z}$ such that $|t/n| < \epsilon$. Then

$$f(t) = f(n(t/n)) = f(t/n)^n = \left(e^{(t/n)A}\right)^n = e^{tA}. \qquad \square$$

If $\rho : G \to GL(V)$ is a representation and $a \in L(G)$, then $t \mapsto \rho(e^{ta})$ is a homomorphism $\mathbf{R} \to GL(V)$. Thus there is a unique $b \in \operatorname{End}(V)$ such that $\rho(e^{ta}) = e^{tb}$.

Definition 2.3.1.2. For $\rho : G \to GL(V)$, define $d\rho : L(G) \to \operatorname{End}(V)$ by

$$\rho(e^{ta}) = e^{td\rho(a)}$$

for all $t \in \mathbf{R}$.

We can also define $d\rho(a)$ by the formula

$$d\rho(a) = \frac{d}{dt}\rho(e^{ta})\bigg|_{t=0}.$$

What are the properties of $d\rho$, and can ρ be recovered from $d\rho$?

Lemma 2.3.1.3. *$d\rho$ is continuous.*

Proof. Let $a \in L(G)$. Pick a bounded neighborhood U of a. Then there exists $t \gg 0$ such that tU is in the domain of log. Thus

$$d\rho(x) = \frac{\log \rho(e^{tx})}{t}$$

for $x \in U$. This formula is a composition of continuous functions and thus is continuous. Hence $d\rho$ is continuous at a for all $a \in L(G)$. ☐

Proposition 2.3.1.4. *i. $d\rho$ is linear and*

$$d\rho[a,b] = [d\rho(a), d\rho(b)].$$

ii. ρ is a C^∞ map.

Proof. i. Certainly $d\rho(\lambda a) = \lambda d\rho(a)$ for $\lambda \in \mathbf{R}$. Now say $a, b \in L(G)$. Let $g_n = e^{a/n}e^{b/n} = e^{\frac{1}{n}(a+b+\alpha_n)}$ for some $\alpha_n \to 0$. Then

$$\begin{aligned}\rho(g_n) &= \rho(\exp(\frac{1}{n}(a+b+\alpha_n))) \\ &= \exp(d\rho(\frac{1}{n}(a+b+\alpha_n))).\end{aligned}$$

Since $d\rho$ is continuous,

$$\lim_{n\to\infty} n \log \rho(g_n) = \lim_{n\to\infty} d\rho(a+b+\alpha_n) = d\rho(a+b).$$

On the other hand, since ρ is a homomorphism,

$$\rho(g_n) = e^{d\rho(a)/n}e^{d\rho(b)/n} = e^{\frac{1}{n}\left(d\rho(a)+d\rho(b)+\alpha'_n\right)},$$

so

$$\lim_{n\to\infty} n \log \rho(g_n) = \lim_{n\to\infty} d\rho(a) + d\rho(b) + \alpha'_n = d\rho(a) + d\rho(b).$$

Thus $d\rho(a+b) = d\rho(a) + d\rho(b)$.

To prove $d\rho[a,b] = [d\rho(a), d\rho(b)]$, use a similar method on $e^{a/n}e^{b/n}e^{-(a+b)/n}$.

ii. Observe exp is a local C^∞-diffeomorphism at 0, and $d\rho$ is linear and thus C^∞. Thus ρ is C^∞ on a neighborhood U of 1. For general $g \in G$, note $\rho(gh) = \rho(g)\rho(h)$, so for $gh \in gU$, we see ρ is C^∞ on gU.

☐

2.3.2. From Lie algebra to Lie group.

LEMMA 2.3.2.1. *Let G be a connected topological group and let U be a neighborhood of* 1*. Then $G = \cup_{n\geq 1} U^n$.*

PROOF. Let $\mathcal{U} = \cup_{n\geq 1} U^n$.

Note that U^n is open: for $g_1 \cdots g_n \in U^n$, $g_1 \cdots g_{n-1} U \subseteq U^n$ is open. Thus U^n is open.

Now we show $\mathcal{U}$ is closed. Suppose $g \in \bar{\mathcal{U}}$. Then every neighborhood of g intersects $\mathcal{U}$. Since the inverse map is a homeomorphism, U^{-1} is open, so $gU^{-1} \cap \mathcal{U}$ is not empty. Thus there are $h \in U$ and $g_1, \ldots, g_k \in U$ such that $gh^{-1} = g_1 \cdots g_k$. Hence $g = g_1 \cdots g_k h \in U^{k+1} \subseteq \mathcal{U}$. Thus $g \in \mathcal{U}$. □

PROPOSITION 2.3.2.2. *Let G be a connected linear group and let $\rho : G \to GL(V)$ be a representation.*

- *i. If $W \subseteq V$, then W is stable under $\rho(G)$ if and only if $d\rho(a)W \subseteq W$ for all $A \in L(G)$.*
- *ii. Let $f : V \to V'$ be a linear map and let $\rho' G \to GL(V')$ be another representation. Then f is G-equivariant if and only if f is $L(G)$-equivariant.*

PROOF. If G is connected linear, then $\exp(L(G))$ contains a neighborhood of 1 by Cartan's theorem. Thus G is generated by $\exp(L(G))$.

- i. If W is stable under $\rho(G)$, then $e^{td\rho(a)}W \subseteq W$ for all $t \in \mathbf{R}$ and $a \in L(G)$. Thus for $w \in W$,

$$d\rho(a)w = \frac{d}{dt} e^{td\rho(a)} w \Big|_{t=0} \in W.$$

  Conversely, if W is stable under $d\rho(L(G))$, then $e^{L(G)}W \subseteq W$. As $e^{L(G)}$ generates G, we find $\rho(G)W \subseteq W$.
- ii. Suppose f is G-linear. Then

$$e^{td\rho'(a)} f = f e^{td\rho(a)}.$$

  Taking the derivative at $t = 0$ gives $d\rho'(a)f = fd\rho(a)$. Conversely, if $d\rho'(a)f = fd\rho(a)$ for all $a \in L(G)$, then $\rho'(e^a)f = f\rho(e^a)$. Thus f commutes with $e^{L(G)}$. As $e^{L(G)}$ generates G, the map f is G-linear.

□

COROLLARY 2.3.2.3. *If V is a representation of connected linear G, then*

$$V^G = \{v \in V \mid L(G)v = 0\}.$$

PROOF. $V^G = \mathrm{Hom}_G(\mathbf{C}, V)$ where $\mathbf{C}$ is the trivial representation *triv* : $G \to GL(\mathbf{C})$, *triv* $= 1$. The differential of *triv* is *dtriv* $= 0$. Thus $f : \mathbf{C} \to V$ is G-equivariant if and only if $d\rho f = f dtriv = 0$. □

DEFINITION 2.3.2.4. A *representation* of $L(G)$ is a linear map $\varphi : L(G) \to \mathrm{End}(V)$ such that $\varphi[a, b] = [\varphi(a), \varphi(b)]$ for all $a, b \in L(G)$.

Proposition 2.3.2.2 tells us that the functor

$$\mathrm{Rep}_G \to \mathrm{Rep}_{L(G)}$$

sending (V, ρ) to $(V, d\rho)$ is fully faithful. However not every representation of $L(G)$ comes from G.

Example 2.3.2.5. Let $G = S^1$. Note that $\pi_1(S^1) \cong \mathbf{Z}$. We view S^1 as the unit circle in $\mathbf{C}^* = GL_1(\mathbf{C})$. Then $L(S^1) = i\mathbf{R}$. A representation $f : L(S^1) \to \operatorname{End}(V)$ is an endomorphism $A \in \operatorname{End}(V)$. This defines a representation $i\mathbf{R} \to GL(V)$ by $ix \mapsto e^{ixA}$. This descends to $S^1 = i\mathbf{R}/2\pi i\mathbf{Z}$ if and only if $e^{2\pi i A} = 1$. It can be shown this occurs only when A is diagonalizable with integer entries.

What really happened is that A defined a representation of the universal cover $\widetilde{S^1} = i\mathbf{R}$, and then we need ρ to vanish on the kernel of the covering map $\widetilde{S^1} \to S^1$.

2.4. (Oct 29) Representations of $SL_2(\mathbf{R})$, SU_2, and $SO_3(\mathbf{R})$.

2.4.1. The linear group of a Lie algebra?

Theorem 2.4.1.1 ([Eti24],11.2). *Suppose G is a linear group, $f : L(G) \to \operatorname{End}(V)$ is linear and preserves the bracket, and G is simply connected. Then there exists a representation $\rho : G \to GL(V)$ such that $d\rho = f$.*

This follows from Lie's three theorems on the existence of certain Lie groups. We won't discuss that. In the main case of interest, $U(n)$, we'll be able to construct representations of the group as needed.

2.4.2. Representations of $SL_2(\mathbf{R})$.

Example 2.4.2.1. Key source of representations: suppose G is a linear group acting smoothly on a manifold X. Then for $a \in L(G)$, we obtain a vector field $\vec{a}$ on X, which at a point x is the tangent vector

$$\vec{a}_x = \frac{d}{dt} e^{ta} x \bigg|_{t=0} \in T_x X.$$

Now when G acts on X, then G also acts on functions $C^\infty(X)$ by $g \cdot f(x) = f(g^{-1})x$. If $V \subseteq C^\infty(X)$ is a subrepresentation, $\rho : G \to GL(V)$, then $d\rho$ is related to the above by

$$d\rho(a) = -\vec{a},$$

since $\frac{d}{dt} f \circ e^{-ta}|_{t=0} = -\vec{a} f$.

Let $G = SL_2(\mathbf{R})$. Then

$$\mathfrak{sl}_2(\mathbf{R}) = L(SL_2(\mathbf{R})) = \{A \in M_2\mathbf{R} \mid \operatorname{tr}(A) = 0\} = .$$

The Lie algebra $\mathfrak{sl}_2 = L(SL_2(\mathbf{R}))$ has basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the relations

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H.$$

Consider $SL_2(\mathbf{R})$ acting on $X = \mathbf{R}^2$. Inside of $C^\infty(X)$, we have the polynomials $\mathbf{C}[x, y]$ in linear coordinates x, y for the plane. $SL_2(\mathbf{R})$ stabilizes $\mathbf{C}[x, y]_m$, the degree m homogeneous polynomials. What is the action of $\mathfrak{sl}_2(\mathbf{R})$ on $\mathbf{C}[x, y]_m$? We may compute the associated vector fields.

$$e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

so

$$\frac{d}{dt} f(e^{tH}(x, y))|_{t=0} = \frac{d}{dt} f(e^t x, e^{-t} y)|_{t=0} = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}.$$

Thus

$$\vec{H} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Similarly

$$e^{tE} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

so

$$\frac{d}{dt}f(e^{tE}(x,y))|_{t=0} = \frac{d}{dt}f(x+ty,y)|_{t=0} = y\frac{\partial f}{\partial x},$$

and so

$$\vec{E} = y\frac{\partial}{\partial x}.$$

Finally,

$$\vec{F} = x\frac{\partial}{\partial y}.$$

Theorem 2.4.2.2. *$\mathbf{C}[x,y]_m$ is a simple representation of $SL_2(\mathbf{R})$ of dimension $m+1$. Every simple representation of $SL_2(\mathbf{R})$ of dimension $m+1$ is isomorphic to $\mathbf{C}[x,y]_m$.*

Proof. By Proposition 2.3.2.2, $\mathbf{C}[x,y]_m$ is simple over $SL_2(\mathbf{R})$ if and only if it is simple over $\mathfrak{sl}_2$.

Let U be the algebra

$$U = \mathbf{C}\langle E,F,H\rangle/([H,E]-2E,[H,F]+2F,[E,F]-H).$$

Then finite-dimensional modules over U are exactly a vector space V equipped with three operators $E,F,H : V \to V$ satisfying the same relations as in $\mathfrak{sl}_2$. By HW 7, problem 1, there is one simple finite-dimensional module for U of dimension $m+1$, characterized by the fact that there is a vector v such that

$$Hv = mv, Ev = 0.$$

(See also [Eti24, p. 11.16]) Note that E acts on $\mathbf{C}[x,y]_m$ by $-y\frac{\partial}{\partial x}$ and H acts on $\mathbf{C}[x,y]_m$ by $y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$. Thus $y^m \in \mathbf{C}[x,y]_m$ satisfies $Ey^m = 0$ and $Hy^m = my^m$. Since $\mathbf{C}[x,y]_m$ has dimension $m+1$, the dimension of the corresponding $\mathfrak{sl}_2(\mathbf{R})$-representation, $\mathbf{C}[x,y]_m$ is a simple representation of $\mathfrak{sl}_2(\mathbf{R})$. Since $SL_2(\mathbf{R})$ is connected, it is also a simple representation of $SL_2(\mathbf{R})$.

Thus $\mathbf{C}[x,y]_m$ is simple. Now let V be a simple representation of $SL_2(\mathbf{R})$ of dimension $m+1$. Then there is an isomorphism $f : \mathbf{C}[x,y]_m \to V$ of $\mathfrak{sl}_2$-representations. Since $SL_2(\mathbf{R})$ is connected, Proposition 2.3.2.2 shows f is also $SL_2(\mathbf{R})$-linear, so $V \cong \mathbf{C}[x,y]_m$ as $SL_2(\mathbf{R})$-representations. □

2.4.3. Representations of SU_2.

Observe

$$\mathfrak{su}_2 = L(SU_2) = \{x \in M_2(\mathbf{C}) \mid \bar{x}^t = -x, tr(x) = 0\}.$$

This has a basis

$$\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

over $\mathbf{R}$, the Pauli matrices (or i times them? I'm not a physicist).

Claim 2.4.3.1. $L(SU_2) \otimes_{\mathbf{R}} \mathbf{C} = L(SL_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C} = \{x \in M_2(\mathbf{C}) \mid tr(x) = 0\} \subseteq M_2(\mathbf{C})$.

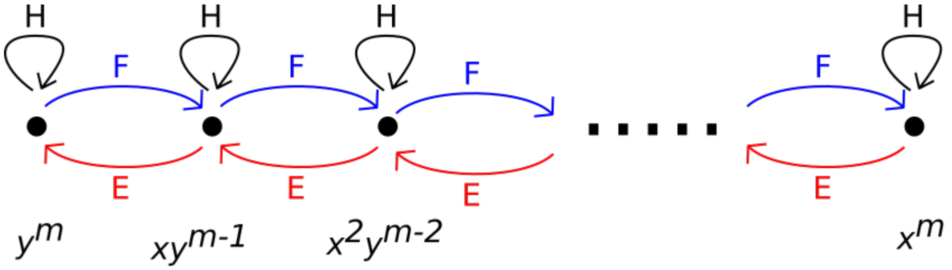

FIGURE 1. The simple representation of $SL_2(\mathbf{R})$ of dimension $m+1$. E and F are "raising" and "lowering" operators.

This can be explicitly written in formulas:

$$H = i\sigma_z \qquad E = \frac{1}{2}\left(i\sigma_x - \sigma_y\right) \qquad F = \frac{1}{2}\left(i\sigma_x + \sigma_y\right)$$

Note that the complex representations of $L(G)$ only depend on $L(G) \otimes_R \mathbf{C}$. Thus

COROLLARY 2.4.3.2. *The representations of* $\mathfrak{su}_2$ *are the same as the representations of* $\mathfrak{sl}_2(\mathbf{R})$.

So, the simple representations of SU_2 are exactly $\mathbf{C}[x,y]_m$. Since SU_2 is compact, this determines all finite-dimensional representations of SU_2. What are the character of these representations?

LEMMA 2.4.3.3. *The character of* SU_2 *on* $\mathbf{C}[x,y]_m$ *is*

$$\chi_m\left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\right) = \lambda^m + \lambda^{m-2} + \cdots + \lambda^{-m} = \frac{\lambda^{m+1} - \lambda^{-m-1}}{\lambda - \lambda^{-1}}.$$

Since every matrix in SU_2 is diagonalizable, to give the character it is enough to give its values on diagonal matrices.

2.4.4. Representations of $SO_3(\mathbf{R})$ and the quaternions. Let $\mathbf{H}$ be the algebra of Hamilton's quaternions: $\mathbf{H}$ has basis $\{1, i, j, k\}$ and algebra structure determined by the relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

This algebra is not commutative since $ij = -ji$. This algebra has a *conjugation*

$$\overline{a_0 + a_1 i + a_2 j + a_3 k} = a_0 - a_1 i - a_2 j - a_3 k.$$

Note that $\overline{ij} = \overline{k} = -k = ji = \overline{j}\,\overline{i}$. Similar computations for the other basis vectors, extended by bilinearity, show that for $z, w \in \mathbf{H}$,

$$\overline{zw} = \overline{w}\,\overline{z}.$$

Thus we have the real part $\Re(z) = \frac{1}{2}(z + \overline{z})$ and imaginary part $\Im(z) = \frac{1}{2}(z - \overline{z})$. The space of imaginary quaternions is three-dimensional, spanned by $\{i, j, k\}$. For $z \in \mathbf{H}$,

$$\overline{z\overline{z}} = \overline{\overline{z}}\,\overline{z} = z\overline{z}.$$

Thus $|z| = \sqrt{z\overline{z}} \in \mathbf{R}$; it is given by the explicit formula

$$|a_0 + a_1 i + a_2 j + a_3 k| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

Every $z \in \mathbf{H} \setminus 0$ is a unit with

$$z^{-1} = \frac{1}{|z|^2}\overline{z}.$$

DEFINITION 2.4.4.1. The group of *norm 1 quaternions* is

$$\mathbf{U} = \{z \in \mathbf{H} \mid |z| = 1\}.$$

$\mathbf{U}$ is a subgroup of $\mathbf{H}^\times$. As a manifold, it is S^3, the unit sphere in $\mathbf{R}^4$.

THEOREM 2.4.4.2. *$\mathbf{U} \cong SU_2$.*

PROOF. The ring $\mathbf{H}$ is not commutative. There is an action of $\mathbf{H}$ on the left and $\mathbf{H}$ on the right; these commute with each other. Consider $\mathbf{C}$ acting on $\mathbf{H}$ by right multiplication by $\mathbf{R} + i\mathbf{R}$. This makes $\mathbf{H}$ into a complex vector space. Consider $\varphi : \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H}) \cong M_2(\mathbf{C})$ given by left multiplication. This map is injective. A quaternion $z = a_0 + a_1 i + a_2 j + a_3 k$ acts in the basis $\{1, j\}$ as follows:

$$\varphi(a_0 + a_1 i + a_2 j + a_3 k) = \begin{pmatrix} a_0 + a_1 i & -a_2 - a_3 i \\ a_2 - a_3 i & a_0 - a_1 i \end{pmatrix}.$$

Thus $\varphi(\overline{z}) = \overline{\varphi(z)}^t$. Note also that $\det(\varphi(z)) = |z|^2$.

For $u \in \mathbf{U}$, $\bar{u} = u^{-1}$, so $\varphi(\mathbf{U}) \subseteq SU_2$. Conversely, if

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in SU_2,$$

then since $\det(x) = 1$,

$$x^{-1} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}.$$

Then $x^{-1} = \overline{x}^t$ implies $x_{22} = \overline{x_{11}}$ and $\overline{x_{21}} = -x_{12}$. Thus $x = \varphi(u)$ for some quaternion $u \in \mathbf{H}$. As $1 = \det(x) = |u|^2$, we conclude $u \in \mathbf{U}$. ☐

2.5. (Oct 31) $SO_3(\mathbf{R})$. The unitary trick.

2.5.1. Representations of $SO_3(\mathbf{R})$. Consider the subspace $\mathbf{R}^3 \cong I \subseteq \mathbf{H}$ of imaginary quaternions; it is three-dimensional. Conjugation of $u \in \mathbf{U}$ preserves conjugation, as

$$\overline{uxu^{-1}} = \overline{u^{-1}}\,\overline{xu} = u\overline{x}u^{-1}.$$

Thus conjugation by $\mathbf{U}$ preserves I.

THEOREM 2.5.1.1. *$\mathbf{U}/\{\pm 1\} \cong SO_3(\mathbf{R})$.*

PROOF. For $x, y \in I$, define an inner product by

$$\langle x, y \rangle = -\Re(xy).$$

In terms of coordinates, this is exactly

$$\langle x_1 i + x_2 j + x_3 k, y_1 i + y_2 j + y_3 k \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

that is, the dot product. I claim that conjugation by u preserves the dot product: if $u \in \mathbf{U}$, then u preserves the product, and u preserves real and imaginary products since conjugation with u commutes with the conjugate. Thus

$$\langle uxu^{-1}, uyu^{-1} \rangle = \langle x, y \rangle.$$

This defines a map

$$\mathbf{U} \to O_3(\mathbf{R}).$$

Since left and right multiplication by $u \in \mathbf{H}$ have determinant $|u|^2$, we see $\mathbf{U} \to O_3(\mathbf{R})$ has image in $SO_3(\mathbf{R})$.

This map is surjective, which can be seen from the following two claims:

Claim 2.5.1.2. *let $u \in U, u = k_0 + v$ for $k_0 \in \mathbf{R}$ and v imaginary. Then there exists unique θ such that $u = \cos(\theta) + \sin(\theta)v'$ where v' is in the unit sphere of imaginary quaternions.*

Claim 2.5.1.3. *The action of $\cos(\theta) + \sin(\theta)v'$ is a rotation of 2θ around v'.*

□

Corollary 2.5.1.4. *There is one simple representation of $SO_3(\mathbf{R})$ of each odd dimension, and no simple representations of even dimension. The simple representations are given by*

$$\begin{tikzcd}
SU_2 \arrow[r] \arrow[d] & GL(\mathbf{C}[x,y]_{2m}) \\
SO_3(\mathbf{R}) \arrow[ur, dashed] &
\end{tikzcd}$$

Proof. If W is a simple representation for $SO_3(\mathbf{R})$, it is also simple after precomposing the action with $SU_2 \to SO_3(\mathbf{R})$. Now the kernel of $SU_2 \to SO_3(\mathbf{R})$ is ± 1. ± 1 acts on the simple SU_2-representation $\mathbf{C}[x,y]_k$ by $(-1)^k$. Thus this representation factors through $SO_3(\mathbf{R})$ if and only if k is even. □

Example 2.5.1.5. $\mathbf{C}[x,y]_2$ is isomorphic to $\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{C}$ as a representation of $SO_3(\mathbf{R})$, where $SO_3(\mathbf{R})$ acts on $\mathbf{R}^3$ by rotations.

The representations $\mathbf{C}[x,y]_{2m}$ of $SO_3(\mathbf{R})$ are known to physicists as *spherical harmonics.*

2.5.2. Reminder on complex analysis.

Let $U \subseteq \mathbf{C}^n$ be open.

Definition 2.5.2.1. $f : U \to \mathbf{C}$ is holomorphic at $x \in U$ if f is C^1 near x and $D_x F : \mathbf{C}^n \to C$ is $\mathbf{C}$-linear.

The condition of $\mathbf{C}$-linearity implies f satisfies the Cauchy-Riemann equations in each variable.

Proposition 2.5.2.2. *f is holomorphic if and only if for all $x \in U$, f is given by a convergent series*

$$f(z) = \sum_{\alpha} c_\alpha (z-x)^\alpha = \sum_{\alpha} c_\alpha (z_1 - x_1)^{\alpha_1} (z_2 - x_2)^{\alpha_2} \cdots$$

near x.

Proposition 2.5.2.3. *If $0 \in U \subseteq \mathbf{C}^n$ is open and connected and $f : U \to C$ is holomorphic, then $f|_{U \cap \mathbf{R}^n} = 0$ implies $f = 0$.*

Proof. The Taylor expansion at 0 can be computed by $f|_{U \cap \mathbf{R}^n}$. Thus $f = 0$ in a neighborhood of 0. Analytic continuation implies $f = 0$ on U. □

2.5.3. Holomorphic linear groups.

DEFINITION 2.5.3.1. A closed subgroup $G \subseteq GL_n(\mathbf{C})$ is *holomorphic* if and only if G is locally given by the zero locus of a set of holomorphic functions.

EXAMPLE 2.5.3.2. $GL_n(\mathbf{C}), SL_n(\mathbf{C}), O_n(\mathbf{C}), Sp_{2n}(\mathbf{C})$ are all holomorphic.

LEMMA 2.5.3.3. *A closed subgroup $G \subseteq GL_n(\mathbf{C})$ is holomorphic if and only if $L(G) \subseteq M_n(\mathbf{C}) = L(GL_n(\mathbf{C}))$ is a $\mathbf{C}$-linear subspace.*

PROOF. First suppose G is holomorphic. Let $a \in L(G)$; then $e^{ta} \in G$ for all $t \in \mathbf{R}$. We want to show $e^{za} \in G$ for all $z \in \mathbf{C}$. Suppose f is a holomorphic function vanishing on G. Then $\varphi(z) = f(e^{za})$ is a holomorphic function on $\mathbf{C}$ vanishing on $\mathbf{R}$. Thus $\varphi = 0$. As this holds for all f, we find $e^{za} \in G$ for all $z \in \mathbf{C}$. Thus $L(G)$ is a $\mathbf{C}$-linear subspace of $M_n(\mathbf{C})$.

Now suppose $L(G)$ is $\mathbf{C}$-linear. Pick a $\mathbf{C}$-linear complement V to $M_n(\mathbf{C})$. Cartan's theorem implies that there is an open neighborhood $U \subseteq M_n(\mathbf{C})$ such that

$$f : L(G) \oplus V \to GL_n(\mathbf{C}),$$
$$f(u \oplus v) = e^u e^v$$

satisfies $f(U) \cap G = f(U \cap L(G))$. Then

$$f(U) \cap G = \{g \in GL_n(\mathbf{C}) \mid \pi(f^{-1}(g)) = 0\}$$

where $\pi : M_n(\mathbf{C}) \to V$ is the projection onto V. These are holomorphic equations since the projection $M_n(\mathbf{C}) \to V$ is $\mathbf{C}$-linear. Thus G is holomorphic at 1. Since G is a group, G is holomorphic. □

EXAMPLE 2.5.3.4. U_n is not holomorphic since $L(U_n)$ is not closed under multiplication by i.

DEFINITION 2.5.3.5. If G is a holomorphic linear group, a *holomorphic representation* is a representation $\rho : G \to GL_n(\mathbf{C})$ which is holomorphic.

THEOREM 2.5.3.6. *If G is a holomorphic linear group, then a representation $\rho : G \to GL_n(\mathbf{C})$ is holomorphic if and only if $d\rho$ is $\mathbf{C}$-linear.*

PROOF. If ρ is holomorphic then $d\rho$ is $\mathbf{C}$-linear. Conversely, on a neighborhood $1 \in U \subseteq G$, $\rho|_U = \exp \circ d\rho \circ \log$, so if $d\rho$ is $\mathbf{C}$-linear, then $\rho|_U$ is holomorphic. Since G is a group, ρ is then holomorphic. □

THEOREM 2.5.3.7 (Weyl). *Holomorphic representations of $GL_n(\mathbf{C})$ are completely reducible. A simple representation of $GL_n(\mathbf{C})$ restricts to a simple representation of U_n.*

PROOF. Let $\mathfrak{gl}_n = L(GL_n(\mathbf{C}))$ and $\mathfrak{u}_n = L(U_n)$. The key of the proof is that "$\mathfrak{u}_n$ is the imaginary axis of $\mathfrak{gl}_n$."

Observe that for $\Theta : GL_n(\mathbf{C}) \to GL_n(\mathbf{C})$ given by $\Theta(g) = (\bar{g}^t)^{-1}$ that $U_n = GL_n(\mathbf{C})^{\Theta}$. If $\theta = d\Theta$, then $\theta(a) = -\bar{a}^t$, so

$$\mathfrak{u}_n = (\mathfrak{gl}_n)^{\theta=1}.$$

Since $\theta^2 = 1$,

$$\mathfrak{gl}_n = (\mathfrak{gl}_n)^{\theta=1} \oplus (\mathfrak{gl}_n)^{\theta=-1} = \mathfrak{u}_n + i\mathfrak{u}_n.$$

Let V be a holomorphic representation of $GL_n(\mathbf{C})$. As U_n is compact, $\mathrm{Res}_{U_n} V$ is completely reducible. If $\mathrm{Res}_{U_n} V$ is simple, we are done. If $\mathrm{Res}_{U_n} V$ is not simple, then there is a decomposition $\operatorname{Res} V = V' \oplus V''$ over U_n. We want to show that V decomposes in the same way for $GL_n(\mathbf{C})$. Let $p \in \operatorname{End}(V)$ be projection onto V'. Then p is U_n-linear. Define $f : \mathfrak{gl}_n \to \operatorname{End}(V)$ by

$$f(a) = \exp(a)p - p\exp(a).$$

Since V is holomorphic, so is f. Since p is U_n-linear, $f(\mathfrak{u}_n) = 0$. Since $\mathfrak{u}_n$ is the imaginary axis of $\mathfrak{gl}_n$, $f = 0$. Thus p is linear over $\exp(\mathfrak{gl}_n)$. Since $GL_n(\mathbf{C})$ is connected, this implies p is $GL_n(\mathbf{C})$-linear, and we are done. $\square$

Left open is the question of which representations of U_n extend to a holomorphic representation of $GL_n(\mathbf{C})$.

2.6. (Nov 5) Representations of $U(n)$

Remark 2.6.0.1. A complex linear group G is *reductive* if it has "compact imaginary part". We list some examples together with their "imaginary part"

- $(SL_n(\mathbf{C}), SU_n)$;
- $(O_n(\mathbf{C}), O_n(\mathbf{R}))$;
- $(Sp_{2n}(\mathbf{C}), Sp_{2n}(\mathbf{C}) \cap U_{2n})$.

There are other definitions of reductive in other settings. They are called reductive because their holomorphic representations are completely reducible.

2.6.1. Matrix entries. For a representation V of G, we have the matrix entries

$$act^* : V^* \otimes V \to C(G),$$

which sends $f \otimes v$ to the functions $g \mapsto f(\rho(g)v)$.

i. If $V = V' \oplus V''$, then $\operatorname{im} act^*_V = \operatorname{im} act^*_{V'} + \operatorname{im} act^*_{V''}$.
ii. If V is simple, then act^*_V is injective;
iii. if V and V' are nonisomorphic simples, then $\operatorname{im} act^*_V$ and $\operatorname{im} act^*_{V'}$ are orthogonal with respect to the Haar integral.

Definition 2.6.1.1. The *ring of representative functions* of a compact group G is

$$F(G) = \sum_V \operatorname{im} act^*_V.$$

Lemma 2.6.1.2. *The ring of representative functions is a ring.*

Proof. If $f \otimes v \in V^* \otimes V$ and $f' \otimes v' \in (V')^* \otimes V'$, then

$$f(gv)f'(gv') = (f \otimes f')(g(v \otimes v')).$$

Thus $act^*(f \otimes v)act^*(f' \otimes v')$ is in the image of the matrix entries for $V \otimes V'$. $\square$

Lemma 2.6.1.3. *i. $F(U_n) = \mathbf{C}[x_{ij}]^n_{i,j=1}[1/\det]$, where $x_{ij} : U_n \to \mathbf{C}$ takes a matrix to its ijth entry.*
ii. Every U_n-representation is a summand of some $(\mathbf{C}^n)^{\otimes k} \otimes \det^{\otimes \ell}$ where $k \geq 0$.

Proof. First note that x_{ij} is a matrix entry of U_n on $\mathbf{C}^n$, and $1/\det$ is a matrix entry of the 1-dimensional representation with character $1/\det$. Since $F(U_n)$ is a ring, $\mathbf{C}[x_{ij}][1/\det]$ maps into $F(U_n)$. This map is injective since functions in $\mathbf{C}[x_{ij}][1/\det]$ are holomorphic, and holomorphic functions on $GL_n(\mathbf{C})$ vanishing on

U_n are 0. Note also that $\mathbf{C}[x_{ij}][1/\det]$ consists of matrix entries of $(\mathbf{C}^n)^{\otimes k} \otimes \det^{\otimes \ell}$ for $k \geq 0$.

Conversely, note that $\mathbf{C}[\Re x_{ij}, \Im x_{ij}]$ is dense in $C(U(n), \mathbf{C})$ by the Stone-Weierstrass theorem. Also $\mathbf{C}[\Re x_{ij}, \Im x_{ij}] = \mathbf{C}[x_{ij}, \bar{x}_{ij}]$. But for $x \in U(n)$, $x^{-1} = \bar{x}^t$, so $\mathbf{C}[x_{ij}, \bar{x}_{ij}] = \mathbf{C}[x_{ij}][1/\det]$. Thus $\mathbf{C}[x_{ij}][1/\det]$ is dense in $C(U_n)$. Since $U(n)$ is compact, this implies $\mathbf{C}[x_{ij}, 1/\det]$ is dense in L^2 norm, so its orthogonal complement is zero. Now suppose that $\operatorname{im} act_W^* \not\subseteq \mathbf{C}[x_{ij}][1/\det]$. Since distinct simples have orthogonal matrix entries, this implies $\operatorname{im} act_W^*$ is orthogonal to $\mathbf{C}[x_{ij}][1/\det]$, so $\operatorname{im} act_W^* = 0$, contradicting that W is a simple representation. Thus $F(U_n) = \mathbf{C}[x_{ij}][1/\det]$.

Since every matrix entry is a matrix entry of some $(\mathbf{C}^n)^{\otimes k} \otimes \det^{\otimes \ell}$, every simple representation appears in such representations. □

As a corollary, we see that every U_n-representation is the restriction of a holomorphic representation of $GL_n(\mathbf{C})$.

EXAMPLE 2.6.1.4. The matrix entries of $U(1)$ are $\mathbf{C}[z, z^{-1}]$. The element z^n corresponds to the irreducible holomorphic representation $z \mapsto z^n : \mathbf{C}^\times \to \mathbf{C}^\times$.

2.6.2. Weight spaces, upper and lower triangular matrices.

In $GL_n(\mathbf{C})$, let T be the diagonal matrices, B be the upper triangular matrices, U the upper triangular matrices with 1's on the diagonal, and U_- the lower triangular matrices with 1's on the diagonal. Note that $B = TU$, and that representations as such products are unique. The Lie algebra $\mathfrak{gl}_n$ has basis E_{ij} for $1 \leq i, j \leq n$. In terms of these, the groups T, B, U, U_- have Lie algebras

$$\mathfrak{t} = span_i\{E_{ii}\}, \mathfrak{u} = span_{i<j}\{E_{ij}\}, \mathfrak{u}_- = span_{i>j}\{E_{ij}\}.$$

If W is a representation of $GL_n(\mathbf{C})$ (now and later holomorphic), its restriction to T decomposes into simple representations of T. Since $T \cong (\mathbf{C}^\times)^n$, the simple representations of T are indexed by $\mathbf{Z}^n$, where

$$\underline{m} \in \mathbf{Z}^n \mapsto \rho_{\underline{m}}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}.$$

So we can write

$$\operatorname{Res}_T W = \bigoplus_{\underline{m} \in \mathbf{Z}^n} W(\underline{m}).$$

We have $E_{ii}|_{W(\underline{m})} = m_i$ since m_i is the differential of $\rho_{\underline{m}}$ on E_{ii}.

CLAIM 2.6.2.1. *If W is a holomorphic representation of $GL_n(\mathbf{C})$, then $E_{ij}W(\underline{m}) \subseteq W(\underline{m} + e_i - e_j)$, where e_i is the ith standard basis vector of $\mathbf{Z}^n$.*

PROOF. Observe that $[E_{kk}, E_{ij}] = (\delta_{ki} - \delta_{kj})E_{ij}$. Thus if $w \in W(\underline{m})$,

$$\begin{aligned} E_{kk}E_{ij}w &= E_{ij}E_{kk}w + [E_{kk}, E_{ij}]w \\ &= m_k E_{ij}w + (\delta_{ki} - \delta_{kj})E_{ij}w, \end{aligned}$$

so $w \in W(\underline{m} + e_i - e_j)$. □

DEFINITION 2.6.2.2. The *dominance order* on $\mathbf{Z}^n$ is defined by $m \geq m'$ if

$$m_1 + \cdots + m_i \geq m'_1 + \cdots + m'_i$$

for all i.

Now $\underline{m} + e_i - e_j > \underline{m}$ if and only if $i < j$. So applying $\mathfrak{u}$ raises weights in dominance order, while applying $\mathfrak{u}_-$ lowers weights in dominance order.

Definition 2.6.2.3. Let W be a holomorphic representation of $GL_n(\mathbf{C})$. A *highest weight vector* is a nonzero $w \in W$ which is fixed by U and is a common eigenvector for T.

Lemma 2.6.2.4. *If $V \neq 0$ is a holomorphic $GL_n(\mathbf{C})$-representation, then V has a highest weight vector.*

Proof. Let $\underline{m}$ be maximal in dominance order such that $V(\underline{m}) \neq 0$. Then if $i < j$, $E_{ij}V(\underline{m}) \subseteq V(\underline{m} + e_i - e_j) = 0$. Thus $v \in V(\underline{m})$ is a weight vector for T and is fixed by U, as desired. $\square$

Corollary 2.6.2.5. *If V is a holomorphic $GL_n(\mathbf{C})$-representation and* $\dim V^U = 1$*, then V is simple.*

Proof. Holomorphic representations of $GL_n(\mathbf{C})$ are completely reducible. If $V = V' \oplus V''$ then $V^U = (V')^U \oplus (V'')^U$. If $\dim V^U = 1$ then one of $(V')^U$ or $(V'')^U$ is zero, which implies that one of V' or V'' is zero. Thus V is simple. $\square$

Definition 2.6.2.6. A weight $\ell \in \mathbf{Z}^n$ is *dominant* if $\ell_i \geq \ell_{i+1}$ for all $i < n$.

Theorem 2.6.2.7 (Highest weight theorem).
- *i. Each simple holomorphic $GL_n(\mathbf{C})$-representation has a unique highest weight vector up to scaling.*
- *ii. Sending a representation to the weight of its highest weight vector is a bijection between simple representations and dominant weights.*

We'll prove this next time.

2.7. (Nov 7) Proof of the highest weight theorem

2.7.1. Examples.

Example 2.7.1.1. Consider $\wedge^k \mathbf{C}^n$. This has weight basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < \cdots < i_k$. If $1 < i_1$ or $i_j + 1 < i_{j+1}$ then applying a suitable E_{ij} for $i < j$ gives another basis vector. Hence $(\wedge^k \mathbf{C}^n)^U = \mathbf{C} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_k$. We conclude that $\wedge^k C^n$ is irreducible with highest weight $\omega_k = (1, \ldots, 1, 0, \ldots, 0)$ with k appearances of 1. This is the kth *fundamental weight* of GL_n.

Example 2.7.1.2. Consider $\mathrm{Sym}^k \mathbf{C}^n$. The vectors fixed by U are exactly scalar multiples of e_1^k. Thus $\mathrm{Sym}^k \mathbf{C}^n$ is simple, and its highest weight is $(k, 0, 0, \ldots) = k\omega_1$.

Example 2.7.1.3. The representation $\bigotimes_{i=1}^n \mathrm{Sym}^{k_i}(\wedge^i \mathbf{C}^n)$ where $k_i \geq 0$ for $i < n$ has a highest weight vector with weight

$$\lambda = (k_1 + \cdots + k_n, k_2 + \cdots + k_n, \ldots, k_n) = k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n.$$

Thus V_λ appears in $\bigotimes_{i=1}^n \mathrm{Sym}^{k_i}(\wedge^i \mathbf{C}^n)$. If $\lambda = k_1\omega_1 + \cdots + k_n\omega_n$, then $k_i = \lambda_i - \lambda_{i+1}$ for $i < n$, and so λ can be written in this form for $k_i \geq 0$ when $i < n$ if and only if λ is dominant.

2.7.2. LU decomposition. An *LU-decomposition* of a matrix a is a decomposition $A = XY$ where X is lower triangular and Y is upper triangular. We will focus on when $X = u_- \in U_-$ and $Y = b \in B$. Note that decompositions u_-b are unique since if $u_-b = u'_-b'$, then $u_-^{-1}u'_- = b(b')^{-1} \in U_- \cap B = \{1\}$.

Define the ith principal minor $f_i(A)$ of a square matrix A to be the determinant of the submatrix with rows and columns in $\{1, 2, \ldots, i\}$.

Lemma 2.7.2.1. *Let k be a field. A matrix $g \in GL_n(k)$ can be written as $u_- b$ for $u_- \in U_-, b \in B$ if and only if $f_i(g) \neq 0$ for all $1 \leq i \leq n$.*

Proof. If $f_1(g) = g_{11} \neq 0$, then by scaling column 1 we can make $g_{11} = 1$, and then by adding row 1 to lower rows and column 1 to later columns, we can clear the first row and column of g. Thus there are $x \in U_-, y \in U$ such that

$$xgy = \begin{pmatrix} g_{11} & 0 \\ 0 & g' \end{pmatrix}$$

where $g' \in GL_{n-1}(k)$. Note that $f_i(g) = f_i(xgy) = g_{11} f_{i-1}(g')$, so $f_{i-1}(g') \neq 0$ for all i. By induction, $g' = x'ty'$ for $x' \in U_-, t \in T, y' \in U$ for GL_{n-1}. Then

$$g = x \begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y' \end{pmatrix} y,$$

so $g \in U_- TU$.

Conversely, if $xty \in U_- TU$, then $f_i(xty) = f_i(t) \neq 0$ for all i. ☐

We used in the proof that if $x \in U_-$ and $y \in U$, then $f_i(xAy) = f_i(A)$. Thus $f_i \in \mathbf{C}[x_{ij}]^{U_- \times U}$.

Corollary 2.7.2.2. *$U_- TU \subseteq GL_n(\mathbf{C})$ is dense.*

Proof. It is the complement of the algebraic hypersurface $f_1 f_2 \cdots f_n = 0$. ☐

Remark 2.7.2.3. The Bruhat decomposition states that for each $g \in GL_n(\mathbf{C})$, there is a unique $\pi \in \Sigma_n$ such that $g = u_- \pi b$ for $u_- \in U_-$ and $b \in B$. The case $\pi = 1$ is the *big cell.*

2.7.3. Proof of highest weight theorem.

Proof of Theorem 2.6.2.7. Write $\mathbf{C}[GL_n] = \mathbf{C}[x_{ij}][1/\det]$, and let $\mathbf{C}[T] = \mathbf{C}[z_i^{\pm 1}]$ be the polynomials on the torus T; z_i is the restriction of x_{ii} to T. As $\mathbf{C}[GL_n]$ is the ring of representative functions, it is the direct sum of $V^* \otimes V$ running over isomorphism classes of simples V, so

$$\mathbf{C}[GL_n]^{U_- \times U} = \bigoplus_{V \text{ simple}} (V^*)^{U_-} \otimes V^U.$$

Since $U_- TU \to GL_n$ is dense, the restriction map $\mathbf{C}[GL_n]^{U_- \times U} \to \mathbf{C}[T] = \mathbf{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is injective. Now

$$f_i|_T = z_1 \cdots z_i,$$

so

$$z_1^{m_1} \cdots z_n^{m_n} = f_1^{m_1 - m_2} f_2^{m_2 - m_3} \cdots f_{n-1}^{m_{n-1} - m_n} f_n^{m_n}.$$

The polynomials $f_1, \ldots, f_n$ are irreducible and pairwise distinct. Since $\mathbf{C}[x_{ij}]$ is a UFD, we have

$$f_1^{m_1 - m_2} \cdots f_{n-1}^{m_{n-1} - m_n} f_n^{m_n} \in \mathbf{C}[x_{ij}][1/\det] \Leftrightarrow m_i \geq m_{i+1} \text{ for all } i.$$

Thus the image is exactly $\mathbf{C}[z_1, \ldots, z_{n-1}, z_n^{\pm 1}]$, so $\mathbf{C}[GL_n]^{U_- \times U} = \mathbf{C}[f_1, \ldots, f_{n-1}, f_n^{\pm 1}]$.

Consider now the action of T on GL_n by right multiplication. Write $X(m)$ for the m-weight space of X with this torus action. Since T normalizes U, T acts on the U-fixed vectors, so

$$\begin{aligned}\mathbf{C}[f_1,\dots,f_{n-1},f_n^{\pm1}](m) &= \mathbf{C}[GL_n]^{U_-\times U}(m)\\ &= \left(\bigoplus_V (V^*)^{U_-}\otimes V^U\right)(m)\\ &= \bigoplus_V (V^*)^{U_-}\otimes V(m)^U.\end{aligned}$$

We have

$$\dim \mathbf{C}[f_1,\dots,f_{n-1},f_n^{\pm1}](m) = \begin{cases}1 & m \text{ is dominant}\\ 0 & \text{else}\end{cases}.$$

Thus

$$\dim \bigoplus_V (V^*)^{U_-}\otimes V(m)^U = \begin{cases}1 & m \text{ dominant}\\ 0 & \text{else.}\end{cases}.$$

We know that $(V^*)^{U_-}$ is always nonzero by Lemma 2.6.2.4. Thus $V(m)^U \neq 0$ only when m is dominant, and if m is dominant, there is a unique V with $V(m)^U \neq 0$, and in this case $\dim V(m)^U = \dim(V^*)^{U_-} = 1$.

Since every representation of GL_n has a highest weight vector, $\dim(V^*)^{U_-} = 1$ for all simple V. By duality, for all simple V, $\dim V^{U_-} = 1$; since U_- and U are conjugate in GL_n, $\dim V^U = 1$ for all simple V. Thus V has a unique highest weight vector up to a scalar.

We have shown assigning V to the weight of V^U is well-defined and injective. The map is surjective since $\mathbf{C}[GL_n]^{U_-\times U}(m) \neq 0$ for dominant m, so a representation with highest weight m exists. ☐

Definition 2.7.3.1. For a dominant weight $\lambda \in \mathbf{Z}^n$, let V_λ be the simple holomorphic representation of GL_n with highest weight λ.

Remark 2.7.3.2 (Borel-Weil theorem). We can construct the representation V_λ of highest weight $\lambda \in \mathbf{Z}^n$ as the left U_--fixed vectors with left T-weight $-\lambda$. That is,

$$\{\phi : GL_n(\mathbf{C}) \to \mathbf{C} \mid \phi \text{ holomorphic}, \phi(b_-g) = f(b_-)\phi(g)\}$$

where $f = f_1^{\lambda_1-\lambda_2}\cdots$. That is, V_λ is the *holomorphic induction* of the character λ of B_- to GL_n. In terms of algebraic geometry, $V_\lambda = H^0(G/B_-, \mathcal{O}(\lambda))$ for a certain line bundle $\mathcal{O}(\lambda)$ on G/B_-.

2.8. (Nov 12) Representations of SL_n. Restriction to GL_{n-1}

2.8.1. More examples of highest weight representations.

Example 2.8.1.1. $(\mathbf{C}^n)^*$ has basis $\{f_1,\dots,f_n\}$ dual to the standard basis $e_1,\dots,e_n$. U has the property that $Ue_i \subseteq e_i + \mathit{span}\{e_1,\dots,e_{i-1}\}$. Dually, $Uf_i \subseteq f_i + \mathit{span}\{f_{i+1},\dots,f_n\}$. Thus $((\mathbf{C}^n)^*)^U = \mathbf{C}f_n$ of weight $(0,\dots,0,-1)$. Thus $(\mathbf{C}^n)^* = V_{(0,\dots,0,-1)}$.

Example 2.8.1.2. $V_\lambda \otimes \det = V_{\lambda+(1,\dots,1)}$. For $\det(U) = 1$, so

$$(V_\lambda \otimes \det)^U = V_\lambda^U \otimes \det,$$

which has torus weight $\lambda + (1,1,\dots,1)$.

Example 2.8.1.3. Combining the above two examples gives that $(\mathbf{C}^n)^* \otimes \det \cong V_{(1,\dots,1,0)} = \wedge^{n-1}\mathbf{C}^n$. How can this be seen? It follows from observing that the wedge product

$$\mathbf{C}^n \otimes \wedge^{n-1}\mathbf{C}^n \to \wedge^n \mathbf{C}^n = \det$$

is a GL_n-equivariant perfect pairing.

2.8.2. GL_n **versus** SL_n**.** For every $g \in GL_n$ and $\lambda \in \mathbf{C}^\times$, $\det(\lambda g) = \lambda^n \det g$. Thus, every matrix in GL_n is the product of a scalar matrix and a matrix of determinant one: $GL_n = \mathbf{C}^\times SL_n$. Such expressions are ambiguous since $\mathbf{C}^\times \cap SL_n = \mu_n$, the group of nth roots of unity (embedded as scalar matrices into SL_n). Thus

$$GL_n = \mathbf{C}^\times \times SL_n/\mu_n.$$

What do representations of SL_n look like? First, we need to think about weights of the torus. Let $\bar{T} = T \cap SL_n$, the diagonal matrices of determinant one. We have a map

$$\mathbf{Z}^n \to \mathrm{Hom}(\bar{T}, \mathbf{C}^\times)$$

coming from restriction from GL_n. The restriction of $(1, 1, \dots, 1)$ is trivial since this corresponds to $diag(z_1, \dots, z_n) \mapsto z_1 \cdots z_n$, which is trivial on $\bar{T}$. Thus if we define

$$\Lambda = \mathbf{Z}^n / \mathbf{Z} \cdot (1, 1, \dots, 1),$$

we find a map $\Lambda \to \mathrm{Hom}(\bar{T}, \mathbf{C}^\times)$. This is an isomorphism, which can be seen by picking an isomorphism $\bar{T} \cong (\mathbf{C}^\times)^{n-1}$.

Definition 2.8.2.1. $\lambda \in \Lambda$ is *dominant* if $\lambda_i \geq \lambda_{i+1}$ for all i. Let Λ_+ be the set of dominant weights.

This is well-defined.

Theorem 2.8.2.2. *The simple holomorphic representations of $SL_n(\mathbf{C})$ are in bijection with Λ_+.*

Proof. Let W be a simple GL_n-representation. Then the center $\mathbf{C}^\times$ acts by scalars on W. If $U \subseteq W$ is SL_n-invariant, then U is also invariant under the action of scalars, so U is GL_n-invariant. Thus W is also simple for SL_n.

Since $V_{\lambda+(1,\dots,1)} = V_\lambda \otimes \det$, and det is trivial on SL_n, we find a well-defined map from Λ_+ to simple SL_n-representations: send λ to $\mathrm{Res}_{SL_n} V_\lambda$.

$$\begin{tikzcd}
\mathbf{Z}^n_+ \arrow[d] & \{\text{simples for } GL_n\} \arrow[l, "\sim"'] \arrow[d, "\mathrm{Res}"] \\
\Lambda_+ \arrow[r, dashed] & \{\text{simples for } SL_n\}
\end{tikzcd}$$

This map is injective: if $\lambda, \lambda' \in \mathbf{Z}_+$ and V_λ and $V_{\lambda'}$ are isomorphic when restricted to SL_n, then $(V_\lambda)^U$ and $(V_{\lambda'})^U$ have the same weight for $\bar{T}$. Thus $\lambda \equiv \lambda'$ mod $(1, \dots, 1)$.

This map is surjective: let $\rho : SL_n \to GL(W)$ be a simple SL_n-representation. Then $Z(SL_n) = \mu_n$ acts by scalars, say $\zeta \mapsto \zeta^m$ for some $m \in \mathbf{Z}/n\mathbf{Z}$. Pick $\tilde{m} \in \mathbf{Z}$ such that $m \equiv \tilde{m} \mod n$, and define $\tilde{\rho} : G \to GL(W)$ by

$$\tilde{\rho}(\lambda g) = \lambda^m \rho(g)$$

where $g \in SL_n$ and $\lambda \in \mathbf{C}^\times$. This is well-defined by our choice of m. □

We see that representation theory for SL_n and GL_n is not so different.

group	GL_n	SL_n	PGL_n
center π_1	$\mathbf{C}^\times$ $\mathbf{Z}$	μ_n 1	1 $\mathbf{Z}/n\mathbf{Z}$
advantage?	connected center	simply connected	simple
Langlands dual	GL_n	PGL_n	SL_n

TABLE 1. SL_n and friends.

2.8.3. Restriction from GL_n to GL_{n-1}.

THEOREM 2.8.3.1. *(GL_n, GL_{n-1}) has simple branching. For $\mu \in \mathbf{Z}^{n-1}$, $\lambda \in \mathbf{Z}^n$ both dominant, $[V_\mu : \operatorname{Res} V_\lambda] = 1$ if and only if*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

A more refined version is restricting not just to GL_{n-1} but to $GL_{n-1} \times GL_1 \leq GL_n$. The simple $GL_{n-1} \times GL_1$-representations are $V_\mu \boxtimes V_m$ for $\mu \in \mathbf{Z}^{n-1}$ dominant and $m \in \mathbf{Z}$.

THEOREM 2.8.3.2. *$(GL_n, GL_{n-1} \times GL_1)$ has simple branching. For $\lambda \in \mathbf{Z}^n$ dominant, $\mu \in \mathbf{Z}^{n-1}$ dominant, and $m \in \mathbf{Z}$, $[V_\mu \boxtimes V_m, \operatorname{Res} V_\lambda] = 1$ if and only if*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}$$

for all i and

$$m = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i.$$

PROOF. Let $U' \subseteq GL_{n-1}$ be the strictly upper triangular matrices in GL_{n-1}, and let $T' \subseteq GL_{n-1}$ be the diagonal matrices. U/U' has coset representatives

$$\left\{ \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbf{C}^{n-1} \right\}.$$

Thus $U_- \backslash U_- TU/U'$ has coset representatives

$$S = \left\{ \begin{pmatrix} t & x \\ 0 & z_n \end{pmatrix} \middle| t \in T', x \in \mathbf{C}^{n-1}, z \in \mathbf{C}^\times \right\}.$$

Restricting to S gives an injective map

$$\mathbf{C}[x_{ij}][1/\det]^{U_- \times U'} \to \mathbf{C}[S] = \mathbf{C}[z_1^\pm, \ldots, z_n^\pm, x_1, \ldots, x_{n-1}].$$

For $g \in GL_n$, define $f_i'(g)$ to be the determinant of the submatrix formed by rows $\{1, 2, \ldots, i\}$ and columns $\{1, 2, \ldots, i-1, n\}$. Then f_i' is invariant under $U_- \times U'$, and on our orbit representatives,

$$f_i' \begin{pmatrix} t & x \\ 0 & z_n \end{pmatrix} = f_{i-1}(t) x_i = z_1 \ldots z_{i-1} x_i.$$

Thus,

$$\mathbf{C}[z_1^\pm, \ldots, z_n^\pm, x_1, \ldots, x_{n-1}] = \mathbf{C}[f_1^\pm, \ldots, f_n^\pm, f_1', \ldots, f_{n-1}'].$$

As before, a monomial in $\{f_1^\pm, \ldots, f_n^\pm, f_1', \ldots, f_{n-1}'\}$ is in the image of $\mathbf{C}[x_{ij}][1/\det]$ if and only if the exponents of f_i are nonnegative for $i < n$. To compute the full image, we need to analyze the weights of $T \times T$ acting on both sides. The weights here are indexed by $\mathbf{Z}^n \times \mathbf{Z}^n$. Observe x_i has weight $(-e_i, e_n)$, so f_i' has weight $(-\omega_i, \omega_{i-1} + e_n)$. I claim that

$$A = \{(-\omega_i, \omega_i)\}_{i=1}^n \cup \{(-\omega_i, \omega_{i-1} + e_n)\}_{i=1}^n$$

is linearly independent in $\mathbf{Z}^n \times \mathbf{Z}^n$. For $(-\omega_i, \omega_{i-1} + e_n) - (-\omega_i, \omega_i) = (0, e_n - e_i)$, and now the claim is evident. Thus each monomial $\{f_1^{\pm}, \dots, f_n^{\pm}, f_1', \dots, f_{n-1}'\}$ has a different weight. Since $\mathbf{C}[x_{ij}][1/\det]^{U_- \times U'}$ is a $T \times T$-representation, it decomposes into weight spaces, and thus

$$\mathbf{C}[x_{ij}][1/\det]^{U_- \times U'} = \mathbf{C}[f_1, \dots, f_n^{\pm}, f_1', \dots, f_n'].$$

To compute $\operatorname{Res} V_\lambda$, we compute the space of vectors with left torus weight $-\lambda$. If $\lambda = k_1\omega_1 + \cdots + k_n\omega_n$, then the possible weight vectors are

$$(-\lambda, (\mu, m)) = \sum_{p_i + q_i = k_i p_i, q_i \geq 0} p_i(-\omega_i, \omega_i) + q_i(-\omega_i, \omega_{i-1} + e_n).$$

Since A is linearly independent, there is a unique such decomposition if one exists. Thus (μ, m) appears in $\operatorname{Res}_{GL_{n-1} \times GL_1} V_\lambda$ if and only if

$$(\mu, m) = \sum_i (p_i\omega_i + q_i\omega_{i-1}) + \sum_i q_i e_n.$$

I claim (μ, m) is of this form if and only if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i and $m = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \mu_i$. For if μ is as above, then

$$\mu_i = \sum_{j=i}^n p_j + \sum_{j=i+1}^n q_j,$$

so

$$\lambda_i = \sum_{j=i}^n (p_j + q_j) \geq \mu_i \geq \sum_{j=i+1}^n (p_j + q_j) = \lambda_{i+1}.$$

Conversely, if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$, then set $p_i = \mu_i - \lambda_{i+1}$ and $q_i = \lambda_i - \mu_i$. □

2.9. (Nov 14) Weyl character formula

2.9.1. Characters. What is the *character* of the representation V_λ with highest weight λ of GL_n?

Definition 2.9.1.1. For $\lambda \in \mathbf{Z}^n$ dominant, $\chi_\lambda \in \mathbf{Z}[z_1^{\pm}, \dots, z_n^{\pm}]$ is the character of V_λ restricted to the diagonal matrices T.

The restriction of the character to T determines the character, since diagonalizable matrices are dense in $GL_n(\mathbf{C})$.

By definition $\chi_\lambda = \sum_{m \in \mathbf{Z}^n} (\dim V_\lambda(m)) z_1^{m_1} \cdots z_n^{m_n}$. It only depends on $\operatorname{Res}_T V_\lambda$.

Remark 2.9.1.2. Theorem 2.8.3.2 gives the following: write $\mu \nearrow \lambda$ if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i, and write $|\lambda| = \sum_i \lambda_i$. Then

$$\chi_\lambda = \sum_{\mu_0 \nearrow \mu_1 \nearrow \cdots \nearrow \mu_n = \lambda} \prod_{i=1}^n z_i^{|\mu_i| - |\mu_{i-1}|}.$$

2.9.2. Weyl group. What is the structure of $\operatorname{Res}_T V_\lambda$?

Definition 2.9.2.1. The *Weyl group* of GL_n is $N(T)/T$.

Proposition 2.9.2.2. *$N(T) = \Sigma_n \cdot T$.*

Proof. Observe that $\mathbf{C}^n$ decomposes into one-dimensional simultaneous eigenspaces for T as $\mathbf{C}\cdot e_1 \oplus \cdots \mathbf{C}\cdot e_n$. Each $g \in N(T)$ sends an eigenspace for T to an eigenspace for T. Thus there is a function $\sigma : [n] \to [n]$ such that $ge_i \in \mathbf{C}e_{\sigma(i)}$. Since g is invertible, σ is a bijection, so $\sigma \in \Sigma_n$. Then $\sigma^{-1}ge_i \in \mathbf{C}e_i$ for all i, which implies $\sigma^{-1}g \in T$. Finally, note that $\Sigma_n \cap T = 1$. □

Corollary 2.9.2.3. *The Weyl group of GL_n is Σ_n.*

For $g \in N(T)$, it follows that g permutes the weight spaces in $\operatorname{Res}_T V$ for any holomorphic GL_n-representation V. Since T preserves the weights, this action factors through $N(T)/T$. Now

$$t\sigma v = \sigma(\sigma^{-1}t\sigma)v,$$

so if $v \in W(\ell)$, then σv is in the weight space $\sigma^{-1}\ell$.

Corollary 2.9.2.4. *χ_λ is a symmetric function: $\chi_\lambda \in \mathbf{Z}[z_1^\pm, \ldots, z_n^\pm]^{\Sigma_n}$.*

Remark 2.9.2.5. Symmetry is not evident in the formula 2.9.1.2.

Example 2.9.2.6. If V has highest weight $\lambda = (m_1, m_2, \ldots)$ then V^* has highest weight $w_0(-\lambda) = (-m_n, -m_{n_1}, \ldots)$, where $w_0(i) = n+1-i$. For in the proof of the highest weight theorem, we see that $(V^*)^{U_-} \otimes V^U$ is spanned by the function f_λ with weight λ for the right torus action. Then f_λ has weight $-\lambda$ for the left torus action, so $(V_\lambda^*)^{U_-}$ has weight $-\lambda$. Further, the involution $w_0(i) = n+1-i$ conjugates U to U_-, so $(V_\lambda^*)^U = \psi(V_\lambda^*)^{U_-}$ has weight $w_0(-\lambda)$, as desired.

The Weyl character formula expresses the character χ_λ, which is symmetric, as a ratio of antisymmetric functions (determinants).

Recall that W has a character $\operatorname{sgn} = \det W|_{\mathfrak{t}}$.

Definition 2.9.2.7. $f \in \mathbf{Z}[z_1^\pm, \ldots, z_n^\pm]$ is *antisymmetric* if $f(wz) = \operatorname{sgn}(w)f(z)$ for all $w \in W$. We write $\mathbf{Z}[z_1^\pm, \ldots, z_n^\pm]^{\operatorname{sgn}}$ for the group of antisymmetric functions.

Definition 2.9.2.8. $A_\lambda(z) = \sum_{w\in\Sigma_n} \operatorname{sgn}(w) z^{w\lambda}$.

By definition, $A_\lambda(z)$ is the determinant

$$A_\lambda(z) = \det \begin{pmatrix} z_1^{\lambda_1} & z_2^{\lambda_1} & \cdots & z_n^{\lambda_1} \\ z_1^{\lambda_2} & z_2^{\lambda_2} & \cdots & z_n^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{\lambda_n} & z_2^{\lambda_n} & \cdots & z_n^{\lambda_n} \end{pmatrix}.$$

Definition 2.9.2.9. $\rho = (n-1, n-2, \ldots, 0) \in \mathbf{Z}^n$.

$A_\rho(z)$ is exactly the Vandermonde determinant $A_\rho(z) = \prod_{i<j}(z_i - z_j)$.

Theorem 2.9.2.10 (Weyl character formula).

$$\chi_\lambda(z) = \frac{A_{\lambda+\rho}(z)}{A_\rho(z)}.$$

Example 2.9.2.11. Consider $G = GL_2$ and let $\lambda = (k, 0)$. Then $\rho = (1, 0)$, so

$$A_{\lambda+\rho}/A_\rho = \frac{z_1^{k+1} - z_2^{k+1}}{z_1 - z_2} = z_1^k + z_1^{k-1} z_2 + \cdots + z_2^k,$$

which is indeed the character of $\mathrm{Sym}^k\, \mathbf{C}^2$. When restricted to SL_2, this recovers formulas we found earlier.

2.9.3. Weyl integral formula.

We will prove the Weyl character formula using orthogonality of characters of $U(n)$, following [Ada69]. To use orthogonality of characters, we need an effective formula for the Haar integral on $U(n)$.

Lemma 2.9.3.1. *Let V be a finite-dimensional vector space and $g \in GL(V)$. Then* $\det(Ad(g)|End(V)) = 1$.

Proof. We may identify $\mathrm{End}(V) \cong V \otimes V^*$ and $Ad(g)$ with $(g \otimes 1)(1 \otimes g^{-1})^*$. Now if V and W are vector spaces and $T : V \to W$, then

$$\det(T \otimes 1) = \det(T)^{\dim W}.$$

Thus

$$\det(Ad(g)) = \det(g)^{\dim V^*} \det((g^{-1})^*)^{\dim V} = 1. \qquad \square$$

Theorem 2.9.3.2. *Let $T_{\mathbf{R}} = (S^1)^n \subseteq U(n)$ be the diagonal unitary matrices. Then there exists a real-valued smooth $u : T_{\mathbf{R}} \to \mathbf{R}$ such that for all class functions f on $U(n)$,*

$$\int_{U(n)} f(g)dg = \int_{T_{\mathbf{R}}} f(z)u(z)dz.$$

Moreover

$$u(z) = \frac{1}{n!} \prod_{i \neq j} (z_i - z_j) = \frac{1}{|W|} |A_\rho(z)|^2.$$

Proof. Write $G = U(n)$ and $T = T_{\mathbf{R}}$ for this proof.

Consider the map $\pi : G/T \times T \to T$ defined by $\pi(g, z) = gzg^{-1}$. Since every unitary matrix is diagonalizable, π is onto. Further, if a unitary matrix s has distinct eigenvalues, then $\pi^{-1}(s)$ is a free W-orbit (an eigenbasis is unique up to permutation and scaling). Hence π has degree $|W|$, so

$$\int_G f(g)dg = \frac{1}{|W|} \int_{G/T \times T} (\pi^* f) \pi^* dg.$$

Note that if f is a class function, then $\pi^* f$ only depends on T and not G/T. We need to express $\pi^* dg$ as a measure on $G/T \times T$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}$ be the Lie algebra of T. We can identify the tangent space of $G/T \times T$ at (g, z) with $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$. The tangent space of G at gzg^{-1} can also be identified with $\mathfrak{g}$ as a vector space. We need to compute

$$D\pi_{g,z} : \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t} \to \mathfrak{g}.$$

For v in a neighborhood of 1 in T

$$\pi(g, zv) = gzvg^{-1} = (gzg^{-1})(gvg^{-1}),$$

so $D\pi_{(g,z)}(v) = \mathrm{Ad}(g)(v)$. For u in a neighborhood of 1 in G/T, $\pi(gu, t) = gutu^{-1}g^{-1}$, so

$$\begin{aligned} D\pi_{g,z}(u) &= guzg^{-1} - gzug^{-1} \\ &= (gzg^{-1})\left((gz^{-1})v(gz^{-1})^{-1} - gug^{-1}\right), \end{aligned}$$

so $D\pi(u) = \mathrm{Ad}(g)(Ad(t^{-1}) - 1)(u)$. Thus

$$\det D\pi = \det(\mathrm{Ad}(g), \mathfrak{g}) \det(\mathrm{Ad}(t^{-1} - 1), \mathfrak{g}/\mathfrak{t}).$$

We can compute both of these determinants after complexifying $\mathfrak{g}$, since determinants are compatible with extending the base field. Now $\det(Ad(g), \mathfrak{gl}_n) = 1$ by Lemma 2.9.3.1. We have $(\mathfrak{g}/\mathfrak{t})_{\mathbf{C}} = \bigoplus_{i \neq j} \mathbf{C}E_{ij}$, and $\mathrm{Ad}(t^{-1}) - 1$ has E_{ij} as an eigenvector with eigenvalue $z_j/z_i - 1$. Now

$$(z_j/z_i - 1)(z_i/z_j - 1) = |z_j/z_i - 1|^2 = |z_i - z_j|^2$$

since z is unitary. Thus

$$u(z) = \frac{1}{n!} \prod_{i<j} |z_j/z_i - 1|^2 = \frac{1}{n!} \prod_{i<j} |z_i - z_j|^2.$$

The Vandermonde determinant says $\prod_{i<j}(z_i - z_j) = A_\rho(z)$. ☐

2.9.4. Proof of Weyl character formula.

Definition 2.9.4.1. $\lambda \in \mathbf{Z}^n$ is *regular* if all parts of λ are pairwise distinct.

Equivalently, λ is regular if the W-stabilizer of λ is trivial.

Lemma 2.9.4.2. *If λ and μ are regular weights, then*

$$\int_T A_\lambda(z)\overline{A_\mu(z)}dz = \begin{cases} \pm|W| & A_\lambda = \pm A_\mu \\ 0 & A_\lambda \neq \pm A_\mu \end{cases}$$

Proof. Note that A_λ and A_μ are sums of W-orbits of characters of T. If λ has nontrivial stabilizer (a repeated entry), then $A_\lambda = 0$; otherwise A_λ is a sum of $|W|$ distinct characters. Thus also either $A_\lambda = \pm A_\mu$ or the sets of characters are disjoint. ☐

Lemma 2.9.4.3. *The functions $\{A_\lambda \mid \lambda$ dominant regular$\}$ form an integral basis for the antisymmetric Laurent polynomials.*

Theorem 2.9.4.4 (Weyl character formula). *If λ is a dominant weight, then $A_\rho\chi_\lambda = A_{\lambda+\rho}$.*

Proof. We can write $A_\rho\chi_\lambda = \sum_i n_i A_{\mu_i}$ for some nonegative integers n_i. By orthogonality of the A_μ's,

$$\int_T |A_\rho\chi_\lambda|^2 = n! \sum_i n_i^2.$$

By the Weyl integral formula,

$$\int_T |A_\rho\chi_\lambda|^2 = n! \int_G |\chi_\lambda|^2 = n!,$$

so $\sum_i n_i^2 = 1$. Thus exactly one n_i is ± 1 and all others are zero. Thus $A_\rho\chi_\lambda = \pm A_\mu$ for some dominant regular μ. The highest weight appearing in the left hand side is $z^\rho z^\lambda$ with coefficient 1, so $\mu = \lambda + \rho$ and $A_\rho\chi_\lambda = A_\mu = A_{\lambda+\rho}$. ☐

2.10. (Nov 19) Schur-Weyl duality

For a statement of Schur-Weyl duality, see [Lor18, §4.7.2].

For the calculation of the highest weight spaces of $\mathrm{Hom}_{\Sigma_n}(L_\lambda, V^{\otimes n})$ where L_λ is a simple Σ_n-representation corresponding to $\lambda \vdash n$, see [Lor18, §8.8].

CHAPTER 3

Modular representations

3.1. (Nov 21) Introduction to modular representation theory

Let k be a field of characteristic $p > 0$. Let G be a finite group. If $p \mid |G|$, then the module theory of kG is not semisimple, for example, the map $kG \to k$ is never split.

Definition 3.1.0.1. A module is *indecomposable* if and only if it is not the direct sum of two submodules.

So we can first break down into direct sums of indecomposable modules, and then ask what the indecomposables are.

Example 3.1.0.2. Let $G = C_p$, the cyclic group of order p. If $x \in C_p$ is a generator, then $kG = k[x]/(x^p - 1)$. In characteristic p,

$$(a+b)^p = a^p + b^p$$

if a and b commute. Thus $0 = x^p - 1 = (x-1)^p$, so $x - 1$ is a nilpotent operator. By Jordan or rational canonical form, every kG module is a direct sum of modules of the form $k[x]/(x-1)^i$ for $1 \leq i \leq p$.

- There is only one simple module, k.
- There are p *indecomposable* modules $k[x]/(x-1)^i$ for $1 \leq i \leq p$.
- Only one indecomposable module is free.

This is the simplest example in modular representation theory.

The "next" example is actually much more complicated:

Example 3.1.0.3. Let $G = C_p \times C_p$. Then there are infinitely many indecomposable kG-modules. See e.g. [Alp86, pp. 27–28]

The answer is to give up on classifying all indecomposables. We many still ask:

i. What are the simple kG-modules? How many are there?
ii. How do the simple kG-modules fit together?

3.1.1. Jordan-Hölder and Krull-Schmidt theorems.

These theorems are our first theorems on how modules are built from simple modules.

Definition 3.1.1.1. A finite *filtration* on a module M is a chain of submodules $0 = F_0M \subseteq F_1M \subseteq \cdots \subseteq F_nM = M$.

Instead of asking for M to decompose into simple modules, we could look for a fitration F such that $F_{i+1}M/F_iM$ is simple.

Theorem 3.1.1.2 (Jordan-Hölder. [Lor18], p. 33). *Let A be a k-algebra and let V be a A-module with $\dim_k V < \infty$. If V has two filtrations $F_\bullet$ and $F'_\bullet$ such that F_i/F_{i-1} and F'_i/F'_{i-1} are simple A-modules for all i, then the multisets*

$$[F_1/F_0, F_2/F_1, \ldots] \qquad \textit{and} \qquad [F'_1/F'_0, F'_2/F'_1, \ldots]$$

are equal.

Proof. We induct on the dimension of V. Suppose $F_\bullet, F'\bullet$ are given, and let $W_\bullet = F_\bullet/F_{\bullet-1}$, $W'_\bullet = F'_\bullet/F'_{\bullet-1}$. Then W_1 and W'_1 are both simple submodules of V. So $W_1 \cap W'_1$ is a submodule of both W_1 and W'_1. Thus either $W_1 = W'_1$ or $W_1 \cap W'_1 = 0$. In the former case, we apply the inductive hypothesis to V/W_1.

Now suppose $W_1 \cap W'_1 = 0$. Then $W_1 \oplus W'_1 \subseteq V$, so let $U = V/(W_1 \oplus W'_1)$. Then U has a composition series $[Z_1, \ldots, Z_p]$. By inductive hypothesis,

$$[W'_1, Z_1, \ldots, Z_p] = [W_2, \ldots, W_n]$$

and

$$[W_1, Z_1, \ldots, Z_p] = [W'_2, \ldots, W'_m].$$

Thus

$$[W] = [W_1, W'_1, Z_1, \ldots, Z_p] = [W'],$$

as desired. ☐

Thus, if A is a finite-dimensional k-algebra and L is a simple module, then sending M to the number of times L appears in a Jordan-Hölder series $\ell_L(M)$ is well-defined.

We have another option on how to break our module into simple pieces: instead of taking subs and quotients of any kind of module, we could try to break into direct sums.

Theorem 3.1.1.3 (Krull-Schmidt. [Lor18], p. 38). *Let A be a k-algebra and V be an A-module such that $\dim_k V < \infty$. Then two decompositions of V into indecomposable summands have the same lists of summands up to isomorphism and reordering.*

Proof. Omitted. It is very similar to the Jordan-Hölder theorem. ☐

3.1.2. Projective modules.

Lemma 3.1.2.1 ([Lor18], §2.1.1). *Let A be a ring and P be an A-module. The following are equivalent:*

- *i. If $f : M \twoheadrightarrow M''$ is a surjective map of A-modules, then every $P \to M''$ lifts to a map $P \to M$:*

$$\begin{tikzcd} P \arrow[d, dashed] \arrow[dr] & \\ M \arrow[r, two heads] & N \end{tikzcd}$$

- *ii. Every surjective $f : M \twoheadrightarrow P$ splits, i.e. admits a right inverse $g : P \to M$ so that $fg = id_P$.*
- *iii. P is a direct summand of a free A-module;*
- *iv. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then*

$$0 \to \mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, M'') \to 0$$

is exact.

Proof. Suppose i. Let $f : M \twoheadrightarrow P$ be surjective. Then the identity map of P lifts along f:

$$\begin{tikzcd}
P \arrow[d, dashed] \arrow[dr, "id"] & \\
M \arrow[r] & P
\end{tikzcd} .$$

Thus i. implies ii.

Now suppose ii. P admits a surjective map from a free module F by choosing some generators. Since the surjection $F \to P$ splits, P is a direct summand of F. Thus ii. implies iii.

Now suppose iii. We know that $\mathrm{Hom}_A(A, -)$ is exact, and taking direct summands preserves exactness. Thus iii. implies iv.

Now suppose iv. If $f : M \twoheadrightarrow M''$ is surjective, then for $M' = \ker(f)$ the sequence $0 \to M' \to M \to M'' \to 0$ is exact. Hence $\mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, M'')$ is surjective, that is, every map from $P \to M''$ lifts through f. Thus iv. implies i. ☐

Definition 3.1.2.2. An A-module is *projective* if it satisfies one of the equivalent conditions of Lemma 3.1.2.1.

It is immediate from Lemma 3.1.2.1 iii. that a summand of a projective module is projective.

How are the simples and the indecomposable projectives related?

Definition 3.1.2.3. A surjection $f : M \to L$ is *essential* if $f(M') \neq L$ for all proper $M' < M$. A *projective cover* of a module L is a projective module P with an essential surjection $P \to L$.

Lemma 3.1.2.4 ([Ser78], §14.3, or [Lor18], p. 89). *If A is a finite-dimensional k-algebra, then*

i. every finitely generated A-module has a projective cover;
ii. sending a simple module to its projective cover is a bijection

$$\{\textit{simples}\}/\cong \leftrightarrow \{\textit{f.g. projective indecomposables}\}/\cong$$

Proof. Let M be a f.g. A-module.

Pick a surjection $F \to M$ from a finite free module, and let $F \to P \to M$ be a maximal quotient of F such that $P \to M$ is essential (exists since F is finite-dimensional). Pick a minimal $Q < F$ such that $Q \twoheadrightarrow P$. Then $Q \to P$ is essential, and $P \to M$ is essential, so $Q \to M$ is also essential.

Since F is free,

$$\begin{tikzcd}
F \arrow[d, dashed, bend right, "a"'] \arrow[dr] & & \\
Q \arrow[u] \arrow[r, two heads] & P \arrow[r, two heads] & M
\end{tikzcd}$$

Then $a(F)$ maps onto P, so since $Q \twoheadrightarrow P$ is essential, $a(F) = Q$ and a is surjective. Since P is a maximal quotient of F which is essential onto M, we find $Q \twoheadrightarrow P$ is an isomorphism. This implies P is a summand of F and so P is projective.

Now we show that projective covers are unique up to isomorphism. Suppose $P \to M$ and $P' \to M$ are both projective covers.

$$\begin{tikzcd} P \arrow[r, dashed, "a"] \arrow[dr, two heads] & P' \arrow[d, two heads] \\ & M \end{tikzcd}$$

Then $a(P) \twoheadrightarrow M$, so $a(P) = P'$. Since P' is projective, a splits, so $P = P' \oplus Q$. Since $P \twoheadrightarrow M$ is essential, $Q = 0$.

Now suppose L is simple. We claim its projective cover is indecomposable. For if $a : P \twoheadrightarrow L$ and $P = P' \oplus P''$ is a proper decomposition, then $a(P')$ and $a(P'')$ are proper submodules of L, so $a(P') = a(P'') = 0$ and $a = 0$. The map $\{\text{simples}\}/\cong \to \{\text{proj. indec.}\}/\cong$ is injective since if P maps onto two different simples L and L', consider kernels K and K'. Since $P \to L'$ is essential, $K' \supseteq K$. Similarly $K \supseteq K'$, so $K = K'$ and thus $L \cong L'$.

If P is indecomposable projective, then P has some simple quotient $P \twoheadrightarrow L$. Let P_L be the cover of L. Then $P \twoheadrightarrow L$ lifts to $P \twoheadrightarrow P_L$, so P_L is a summand of P. Since P is indecomposable, $P_L \cong P$. Thus the map is surjective. ☐

Corollary 3.1.2.5. *If A is a finite-dimensional k-algebra, then*

i. Every indecomposable projective is finitely generated.
ii. There are finitely many simple A-modules.

Proof. If P is an indecomposable projective, then P has a simple quotient L. If P_L is the projective cover of L, then $P \twoheadrightarrow P_L$, so P_L is a summand of P. Thus $P \cong P_L$.

There are finitely many f.g. indecomposable projectives since they are summands of A, and there are finitely many such due to the Krull-Schmidt theorem. ☐

3.2. (Nov 26) Reduction modulo p

3.2.1. Grothendieck groups. Cartan homomorphism.

Definition 3.2.1.1. Let A be a finite-dimensional k-algebra. Define $K_0(A)$ to be the quotient of the free abelian group on all finitely generated A-modules by the relations $[M] = [M/M'] + [M']$ whenever $M' \subseteq M$ is a submodule of M.

The Jordan-Hölder theorem implies that $K_0(A)$ is free abelian with basis the simple modules for A.

Definition 3.2.1.2. Let A be a finite-dimensional k-algebra. Define $K^0(A)$ to te the quotient of the free abelain group on all f.g. *projective* A-modules, modulo the relations $[P] = [P'] + [P'']$ whenever $P = P' \oplus P''$.

The Krull-Schmidt theorem implies that $K^0(A)$ is free abelian on the indecomposable projectives for A.

Remark 3.2.1.3. Lemma 3.1.2.4 implies $K_0(A)$ and $K^0(A)$ are free abelian of the same rank. However, sending simple $[L]$ to $[P_L]$ is not natural as a functor in A, so the functors K_0 and K^0 are not naturally isomorphic on finite-dimensional algebras.

Since every projective module is a module, and every short exact sequence of projectives splits, we have a map

$$c : K^0(A) \to K_0(A)$$

defined by $c[P] = [P]$, called the *Cartan homomorphism* [Ser78, §15.1].

EXAMPLE 3.2.1.4. For $A = kC_p$ where $\operatorname{char} k = p$, the Cartan homomorphism has matrix $[p]$: there is one simple module k with cover $k[x]/(x-1)^p$, so $K_0(kC_p) = \mathbf{Z} \cdot [k]$, $K^0(kC_p) = \mathbf{Z} \cdot [kC_p]$. The filtration $F_i = (x-1)^{p-i}k[x]/(x-1)^p$ has $F_i/F_{i-1} \cong k$, and there are p of them. Thus $c[kC_p] = p[k]$.

3.2.2. Reduction modulo p. [Ser78, §15.2]. Our approach to modular representation theory will come from comparing characteristic 0 and characteristic p.

DEFINITION 3.2.2.1. A *p-modular system* is a discrete valuation ring $(R, \mathfrak{m})$ such that $k = R/\mathfrak{m}$ is algebraically closed of characteristic p and $K = \operatorname{Frac}(R)$ is of characteristic 0.

EXAMPLE 3.2.2.2. Let $R = \widehat{\mathbf{Z}_p(\zeta_{p'})}$ be the ring formed by adjoining all the roots of unity of order prime to p to $\mathbf{Z}_p$, then p-adically completing. Then R is a discrete valuation ring with maximal ideal pR, and $R/pR \cong \bar{\mathbf{F}}_p$. $K = \widehat{\mathbf{Q}_p(\zeta_{p'})}$.

EXAMPLE 3.2.2.3. If k is any algebraically closed field of characteristic p, then a p-modular system exists: $R = W(k)$ is the Witt vector ring over k, and $\mathfrak{m} = pW(k)$.

EXAMPLE 3.2.2.4. If $(R, \mathfrak{m}, k, K)$ is a p-modular system and L/K is a finite extension, then $\mathcal{O}_L$ also defines a p-modular system with the same residue field; however $\mathfrak{m}_L \neq \mathfrak{m}m\mathcal{O}_L$, for k being algebraically closed implies L/K is totally ramified.

Fix a p-modular system $(R, \mathfrak{m}, k = R/\mathfrak{m}, K = \operatorname{Frac}(R))$. Then we have maps

$$kG \leftarrow RG \to KG,$$

which we can use to compare characteristic p and characteristic zero.

DEFINITION 3.2.2.5. A *lattice* in a K-vector space M is a finitely-generated R-submodule M_1 such that $K \otimes_R M_1 = M$.

LEMMA 3.2.2.6. *Let M be a finite-dimensional KG-module. Then there is an R-lattice $M_1 \subseteq M$ stable under G.*

PROOF. Let M_0 be some lattice stable under R. Define $M_1 = \cup_{g \in G} gM_0$. This is again a lattice and is stable under G. □

Thus, if M is a KG-module, I can get a kG-module from $M_1/\mathfrak{m}M_1$.

LEMMA 3.2.2.7. *[Ser78, §15.2] Let $(R, \mathfrak{m}, k, K)$ be a p-modular system. Let M_1 and M_2 be two G-stable lattices in a finite-dimensional KG-module M. Then $[M_1/\mathfrak{m}M_1] = [M_2/\mathfrak{m}M_2]$ in $K_0(kG)$.*

PROOF. First suppose that $\mathfrak{m}M_1 \subseteq M_2 \subseteq M_1$. Then M_1/M_2 is a kG-module, and we have a four-term exact sequence of kG-modules:

$$0 \to \mathfrak{m}M_1/\mathfrak{m}M_2 \to M_2/\mathfrak{m}M_2 \to M_1/\mathfrak{m}M_1 \to M_1/M_2 \to 0.$$

Since $\mathfrak{m}$ is principal, $\mathfrak{m}M_1/\mathfrak{m}M_2 \cong M_1/M_2$ as kG-modules. Thus $[M_1/\mathfrak{m}M_1] = [M_2/\mathfrak{m}M_2]$ in this case.

Now in general, we can assume by rescaling that $M_2 \subseteq M_1$. Since M_2 is a lattice, $\mathfrak{m}^n M_1 \subseteq M_2$ for some n. Induct on n; the base case $n = 1$ was established above. If $M_3 = \mathfrak{m}^{n-1}M_1 + M_2$, then

$$\mathfrak{m}^{n-1}M_1 \subseteq M_3 \subseteq M_1$$

and

$$\mathfrak{m}M_3 \subseteq M_2 \subseteq M_3.$$

Thus $[M_1/\mathfrak{m}M_1] = [M_3/\mathfrak{m}M_3] = [M_2/\mathfrak{m}M_2]$ by induction. $\square$

Definition 3.2.2.8. Define the *decomposition homomorphism* $d : K_0(KG) \to K_0(kG)$ by sending $[M]$ to $[M_1/\mathfrak{m}M_1]$ for any lattice $M_1 \subseteq M$.

3.3. (Dec 03) Lifting. Brauer characters

Example 3.3.0.1. Last time, I promised an example of a KG-module M with two lattices M_1, M_2 such that $M_1/\mathfrak{m}M_1$ and $M_2/\mathfrak{m}M_2$ are different kG-modules.

Assume that a primitive pth root of unity $\zeta_p \in R$ and take $G = C_p = \langle x \rangle$. Consider $M = KC_p$ the regular representation. Then $M_1 = RC_p$ has reduction $M_1/\mathfrak{m}M_1 = kC_p$, which is indecomposable of dimension p.

On the other hand, let $e_1, \ldots, e_p \in KC_p$ be the idempotents corresponding to the p simple characters over K; that is

$$e_i = \sum_{j=0}^{p-1} \zeta_p^{ij} x^j.$$

Then $M_2 = Re_1 + \cdots + Re_p$ is a lattice in KC_p. Each Re_i is stable under C_p. Since $(\zeta_p - 1)^p \equiv \zeta^p - 1^p = 0 \mod p$, we have $p \mid (\zeta_p - 1)^p$, so $\zeta_p - 1 \in \mathfrak{m}$. Then $Re_i/\mathfrak{m}e_i = k$ is the trivial representation of C_p, so $M_2 = \bigoplus_{i=1}^p k$ is p copies of the trivial representation.

3.3.1. Lifting.

Lemma 3.3.1.1 (Noncommutative Hensel's Lemma). *Let R be a commutative ring. Let $f \in R[x]$ such that $(f', f) = (1) \subseteq R[x]$. If A is an R-algebra complete with respect to an ideal I, and $x_0 \in A/I$ satisfies $f(x_0) = 0$, then*

- i. *there exists $x \in A$ such that $x \equiv x_0 \mod I$ and $f(x) = 0$;*
- ii. *such a lift is unique up to conjugation by $1 + I$.*

Proof. Since A is complete, the general statement reduces to the case when $I^2 = 0$. So we may assume that $I^2 = 0$. Note that the hypothesis $(f, f') = 1$ implies $f'(x_0)$ is invertible in A/I.

First, we show that a lift x such that $f(x) = 0$ exists. Suppose that $\tilde{x}$ is any lift of x_0. $f'(\tilde{x})$ is invertible in A since $f'(x_0)$ is invertible in A/I. Set

$$x = \tilde{x} - f'(\tilde{x})^{-1} f(\tilde{x}).$$

Note that $f(\tilde{x}) \in I$. Since $\tilde{x}$ commutes with any polynomial in $\tilde{x}$, and $I^2 = 0$, we have by Taylor expansion

$$f(x) = f(\tilde{x}) + f'(\tilde{x}) \left(-f'(\tilde{x})^{-1} f(\tilde{x})\right) = 0.$$

Thus a desired lift x exists.

Now suppose that y is another lift of x_0 to a zero of f. Set $h = y - x$; if $u = 1 + v$ for $v \in I$, then

$$uxu^{-1} = x + [v, x],$$

so the goal is to show that $h = [v, x]$ for some $v \in A$. This claim depends only on h and x, so we may replace A with the subalgebra A_0 generated by h and x. This algebra is spanned by expressions x^α and $x^\alpha h x^\beta$, as h is in a two-sided square-zero ideal. Then $J = [x, A_0]$ is a two-sided ideal in A_0: $[x, x] = 0$ so $[x, A_0] \subseteq I$; $[x, A_0]$ is closed under multiplication by x on both sides, and also by multiplication by h on both sides since $hI = Ih = 0$. The ring A_0/J is commutative, so

$$0 = f(x + h) - f(x) = f'(x)h \mod J.$$

As $f' \in R[t]/(f)$ is a unit, $f'(x)$ is invertible in A_0/J, so $h = 0 \mod J$. Thus $h \in [x, A_0]$, as desired. $\square$

Applying Hensel's lemma to idempotents allows us to lift projective modules from characteristic p to characteristic 0.

Corollary 3.3.1.2. *Let P be a finitely generated projective kG-module. Then there is a projective RG-module $\tilde{P}$ such that $\tilde{P}/\mathfrak{m}\tilde{P} \cong P$, unique up to an isomorphism congruent to* 1 *modulo* $\mathfrak{m}$.

Proof. Let F be a finitely generated free kG-module containing P as a summand. Then there exists $e \in \mathrm{End}_{kG}(F)$ such that $e^2 = e$ and $P = eF$. Note that $\mathrm{End}_{kG}(kG) = kG^{op}$ by right multiplication so $\mathrm{End}_{kG}(F)$ is a matrix algebra over $(kG)^{op}$.

Now let $\tilde{F}$ be a lift of F; then $A = \mathrm{End}_{RG}(\tilde{F})$ is a matrix algebra over RG and so is complete with respect to $I = \mathfrak{m}A$. Let $f(x) = x^2 - x$; then $f'(x) = 2x - 1$ is coprime to f since

$$(2x - 1)^2 - 4(x^2 - x) = 4x^2 - 4x + 1 - 4x^2 + 4x = 1.$$

Thus there is $\tilde{e} \in \mathrm{End}_{RG}(\tilde{F})$ lifting e such that $\tilde{e}^2 = \tilde{e}$, unique up to conjugation. $\tilde{P} = \tilde{e}\tilde{F}$ is our lift. Uniqueness up to conjugation implies uniqueness up to isomorphism: if $\tilde{P}'$ is another lift, then $\tilde{P}'$ can also be made a summand of $\tilde{F}$. $\square$

After tensoring with $K = \mathrm{Frac}(R)$, we can lift a f.g. projective kG-module to a KG-module.

Definition 3.3.1.3. Define $e : K^0(kG) \to K^0(KG)$ by sending $[P]$ to $[\tilde{P} \otimes_R K]$.

Theorem 3.3.1.4 (cde triangle). *The following triangle commutes:*

$$\begin{tikzcd}
K^0(kG) \arrow[rr, "c"] \arrow[dr, "e"'] & & K_0(kG) \\
& K^0(KG) = K_0(KG) \arrow[ur, "d"'] &
\end{tikzcd},$$

i.e. $c = de$.

Proof. Suppose that P is a f.g. projective kG-module, a summand of $F = (kG)^n$. Let $\tilde{P}$ be a lift which is a summand of $\tilde{F} = (RG)^n$. Then $e[P] = [\tilde{P} \otimes_R K]$. But $\tilde{P} \otimes_R K$ has a lattice $\tilde{P}$, and $\tilde{P}/\mathfrak{m}\tilde{P} \cong P$ by definition. Thus the triangle commutes. $\square$

3.3.2. Brauer characters.

Definition 3.3.2.1. Let G be a finite group and p be a prime number. $g \in G$ is *p-regular* if the order of g is prime to p. $g \in G$ is *p-unipotent* if the order of g is a power of p.

Proposition 3.3.2.2 (Jordan decomposition). *If G is a finite group and p is a prime number, then every $g \in G$ can be written in a unique way as $g = g_r g_u$ where g_r, g_u are p-regular and p-unipotent and $g_r g_u = g_u g_r$.*

Proof. Decompose the order of g as mp^e where $p \nmid m$, and write $1 = sp^e + tm$. Then $g_r = g^{sp^e}$ and $g_u = g^{tm}$ is one such decomposition. If $g = g'_r g'_u$ is another decomposition, then $g^m = (g'_r)^m (g'_u)^m$ has order a power of p, so $(g'_r)^m = 1$ and $g'_u = g_u$; thus $g'_r = g_r$. □

It turns out that ordinary characters only see the p-regular part of a group element.

Lemma 3.3.2.3. *If V is a finite-dimensional kG-module, then*

$$tr(g|V) = tr(g_r|V).$$

Proof. We may assume $G = \langle g \rangle$. Decompose $V = \bigoplus_\zeta V(\zeta)$ into eigenspaces for g_r. Since g_u commutes with g_r, g_u acts on each $V(\zeta)$, and

$$tr(g|V) = \sum_\zeta \zeta tr(g_u|V(\zeta)).$$

Thus it suffices to show $tr(g_u|V(\zeta)) = \dim V(\zeta)$. Since $g_u^{p^e} = 1$ for some e, the possible minimal polynomials of Jordan blocks are $(g_u - 1)^k$ for $k \leq p^e$ (i.e. g_u is unipotent). Hence the only eigenvalue of g_u is 1, so the trace of g_u equals the dimension. □

Theorem 3.3.2.4 (Brauer). *The number of simple kG-modules up to isomorphism is equal to the number of p-regular conjugacy classes.*

To prove Brauer's theorem, we need to enhance ordinary characters.

Definition 3.3.2.5. Let V be a finite-dimensional k-vector space, and let $\tilde{V}$ be a free R-module such that $\tilde{V} \otimes_R k \cong V$. Suppose $g \in \operatorname{End}_k(V)$ has finite order prime to p. The *Brauer trace* of g is defined to be

$$\operatorname{tr}_{Br}(g|V) = \operatorname{tr}(\tilde{g}|\tilde{V}) \in R \subseteq K,$$

where $\tilde{g} \in \operatorname{End}_R(\tilde{V})$ is any lift of g with the same order.

Since the polynomial $f(t) = t^n - 1$ is separable when $p \nmid n$, Lemma 3.3.1.1 applies to show that $g \in \operatorname{End}_k(V)$ has a lift which is unique up to conjugation.

Remark 3.3.2.6. Brauer's original definition of $\operatorname{tr}_{Br}(g|V)$ was to diagonalize g, then lift the eigenvalues to R. In fact, he took an isomorphism of the roots of unity prime to p in k with those in $\mathbf{C}$, and viewed the Brauer trace as a complex number. I think the Brauer trace is more naturally a p-adic number.

Let G_{reg} denote the set of p-regular elements of G, and let $Cl(G_{reg}, A)$ denote the A-valued class functions on G_{reg}.

Definition 3.3.2.7. The *Brauer character* of a kG-module V is defined on p-regular $g \in G_{reg}$

$$\phi_V(g) = \mathrm{tr}_{Br}(g|V).$$

We have $\phi_V \in Cl(G_{reg}, K)$.

If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of kG-modules, then $\phi_V = \phi_{V'} + \phi_{V''}$, as trace is additive in short exact sequences. Thus we have a map $\phi : K_0(kG) \to Cl(G_{reg}, K)$. What we are actually going to show is that ϕ induces an isomorphism

$$\phi : K_0(kG) \otimes_{\mathbf{Z}} K \to Cl(G_{reg}, K).$$

In other words, the Brauer characters of simples form a basis for the class functions on G_{reg}.

Remark 3.3.2.8. If $V \in K_0(KG)$, then $\phi_{dV} = \chi_V|_{G_{reg}}$. In other words, the Brauer character of a reduction modulo p is just the ordinary character restricted to G_{reg}.

Proposition 3.3.2.9. *If P is a projective RG-module and $g \in G$ is not p-regular, then* $\mathrm{tr}(g|P) = 0$.

Proof. For $g \in G$, write $g = g_r g_u$ for the Jordan decomposition; since g is not p-regular, $g_u \neq 1$. Let $C = \langle g \rangle$ be the cyclic subgroup generated by g. Then $\mathrm{Res}_C^G P$ is a projective RC-module. We can decompose P into eigenspaces P_ζ for g_r:

$$P = \bigoplus_\zeta P_\zeta.$$

Each P_ζ, being a summand of P, is a projective C-module and thus a projective $C_u = \langle g_u \rangle$-module. Now $kC_u = k[g_u]/(g_u^{p^e} - 1)$, so its only projective module is free by Jordan decomposition. If Q is a projective RC_u-module, then $Q/\mathfrak{m}Q$ is free, which in turn implies that Q is free since lifts of projectives are unique up to isomorphism (Corollary 3.3.1.2). Thus P_ζ is a free RC_u-module for all ζ. The trace of g_u on the regular representation is zero as long as $g_u \neq 1$. Then

$$\mathrm{tr}(g|P) = \sum_\zeta \zeta \, \mathrm{tr}(g_u|P_\zeta) = \sum_\zeta \zeta \cdot 0 = 0. \qquad \square$$

3.4. (Dec 05) Proof of Brauer's theorem. Blocks

3.4.1. Proof of Brauer's theorem.

Theorem 3.4.1.1. *Let P be a projective kG-module and V a kG-module.*

i. $\dim_k P^G = \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_P(g)$.

ii. $\dim_k \mathrm{Hom}_{kG}(P, V) = \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_P(g^{-1}) \phi_V(g)$.

Proof. Let $\tilde{P}$ be a lift of P to a projective RG-module. Then by Proposition 3.3.2.9,

$$\chi_{\tilde{P}}(g) = \begin{cases} \phi_P(g) & g \in G_{reg} \\ 0 & g \notin G_{reg} \end{cases}.$$

By Theorem 1.3.1.5 part i,

$$\dim_K(\tilde{P} \otimes_R K)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{\tilde{P}}(g) = \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_P(g).$$

Now how are $(\tilde{P} \otimes_R K)^G$ and P^G related? For a free RG-module F, F^G is a free R-module, and $(F \otimes_R K)^G = F^G \otimes_R K$ and $(F/\mathfrak{m}F)^G = F^G/\mathfrak{m}F^G$. Thus $\dim_k(F/\mathfrak{m}F)^G = \dim_k(F \otimes_R K)^G$. Both of $(F/\mathfrak{m}F)^G$ and $F^G/\mathfrak{m}F^G$ are additive in direct sums, and similarly for $- \otimes_R K$, so we conclude for a general projective $\tilde{P}$ that

$$\dim_k(\tilde{P}/\mathfrak{m}\tilde{P})^G = \dim_K(\tilde{P} \otimes_R K)^G.$$

Now consider $\mathrm{Hom}_k(P, V)$ as a kG-module. If P is a summand of a free module F, then $\mathrm{Hom}_k(P, V)$ is a summand of $\mathrm{Hom}_k(F, V)$. Now $\mathrm{Hom}_k(kG, V) \cong \underline{V} \otimes_k kG$ is a free kG-module, so $\mathrm{Hom}_k(P, V)$ is a projective kG-module. The Brauer character of $\mathrm{Hom}_k(P, V)$ is $g \mapsto \phi_P(g^{-1})\phi_V(g)$. Thus applying i. to $\mathrm{Hom}_k(P, V)$ gives ii. $\square$

LEMMA 3.4.1.2. *Let K be a field of characteristic zero and G be a finite group. Then there is a finite extension K'/K such that the characters of $K'G$-modules span $Cl(G, K')$.*

PROOF. Fix an algebraic closure $\bar{K}$ of K, and let K' be the extension of K by all character values of simple $\bar{K}G$-modules. Then K' is a finite extension, and if χ is a simple $\bar{K}G$-character, then

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g \in K'G.$$

Now $e_\chi K'G$ is a $K'G$-module, and its base change to $\bar{K}$ is $\mathrm{End}(V_\chi)$, so its character is $\chi(1)\chi$. Now $\{\chi(1)\chi\}_\chi$ spans $Cl(G, \bar{K})$ and is defined over K', so it spans $Cl(G, K')$. $\square$

REMARK 3.4.1.3. In fact, Brauer proved that all of the simple $\bar{K}G$-modules are defined over $K(\zeta)$ where ζ is a primitive $|G|$th root of unity [Ser78, §12.3].

THEOREM 3.4.1.4. *Let k be an algebraically closed field of characteristic p and $(R, \mathfrak{m}, k, K)$ be a p-modular system with residue field k. Then the Brauer character map $\phi : K_0(kG) \otimes_{\mathbf{Z}} K \to Cl(G_{reg}, K)$ is an isomorphism.*

PROOF. First, we show the map ϕ is injective. Let $L_1, \ldots, L_n$ be the isomorphism classes of simple modules; we need to show $\{\phi_{L_1}, \ldots, \phi_{L_n}\}$ is linearly independent. Let the projective covers of the simples be $P_1, \ldots, P_n$. Then

$$\dim_k \mathrm{Hom}_{kG}(P_i, L_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

By Theorem 3.4.1.1,

$$\dim_k \mathrm{Hom}_{kG}(P_i, L_j) = \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_{P_i}(g^{-1})\phi_{L_j}(g).$$

Thus, if $\sum_j c_j \phi_{L_j} = 0$, then taking the product with ϕ_{P_i} shows $c_i = 0$ for all i.

Now we show ϕ is surjective. It suffices to prove that $\phi \otimes_K K'$ is surjective for some finite extension K'/K. By Lemma 3.4.1.2, there is a finite extension K'/K so that the characters of ordinary $K'G$-modules span $Cl(G, K')$. The p-modular system $(R, \mathfrak{m}, k, K)$ has an extension to a p-modular system $(R', \mathfrak{m}', k, K')$ with the same residue field, as k is algebraically closed. So suppose $f \in Cl(G_{reg}, K')$. Then there is $V \in K_0(K'G) \otimes_{\mathbf{Z}} K'$ such that χ_V agrees with f on G_{reg}. Then $dV \in K_0(kG) \otimes_{\mathbf{Z}} K'$ has

$$\phi_{dV} = \chi_V|_{G_{reg}} = f.$$

Thus ϕ is surjective. $\square$

Corollary 3.4.1.5. *If k is an algebraically closed field, then the number of simple kG-modules is equal to the number of p-regular conjugacy classes.*

Remark 3.4.1.6. The proof also shows that if $p \nmid |G|$, then the decomposition map $d : K_0(KG) \to K_0(kG)$ is an isomorphism for sufficiently large K and k. Thus the representation theory "is the same."

3.4.2. Examples of Brauer tables and decomposition matrices. The following is quite useful even for the basic examples.

Lemma 3.4.2.1 ([Ser78],§16.4). *Let L be a simple $\bar{K}G$-module defined over K. Suppose* $\dim L$ *is divisible by the highest power of p divising $|G|$. Then:*

i. if $L_1 \subseteq L$ is a G-stable lattice, then L_1 is a simple projective RG-module;
ii. $L_1/\mathfrak{m}L_1$ is a simple projective kG-module.

Example 3.4.2.2. Let $G = \Sigma_3$. The ordinary character table is:

	1	(12)	(123)
χ_{tr}	1	1	1
χ_{alt}	1	-1	1
χ_{std}	2	0	-1 .

We are only concerned with the primes 2 and 3. The 2-regular classes are 1 and (123), while the 3-regular classes are 1 and (12).

At $p = 2$, Lemma 3.4.2.1 implies χ_{std} restricts to a simple Brauer characte ϕ_{std}. Thus the table is: Note that $\chi_{alt} \equiv \chi_{tr} \mod 2$. The decomposition matrix at $p = 2$ is thus $D_2 = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$.

$p = 2$	1	(123)
ϕ_{tr}	1	1
ϕ_{std}	2	-1

At $p = 3$, the two one-dimensional characters restrict to all the simples. Note $d\chi_{std} = \phi_{tr} + \phi_{alt}$. The decomposition matrix at $p = 3$ is $D_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

$p = 3$	1	(12)
ϕ_{tr}	1	1
ϕ_{alt}	1	-1

Example 3.4.2.3. Let $G = \Sigma_4$. The ordinary character table is:

	1	(12)	(123)	(12)(34)	(1234)
(4)	1	1	1	1	1
(3,1)	3	1	0	-1	-1
(2,2)	2	0	-1	2	0
(2,1,1)	3	-1	0	-1	1
(1,1,1,1)	1	-1	1	1	-1

At $p = 2$, the only 2-regular classes are 1 and (123). There is only one simple Brauer character of dimension 1, the trivial character. The reduction of $\chi_{(2,2)}$ is not the sum of two 1d characters since it is not $2\phi_{(4)}$. Thus $d\chi_{(2,2)}$ is a simple Brauer character. The Brauer table at $p = 2$ is

$p = 2$	1	(123)
$d\chi_{(4)}$	1	1 .
$d\chi_{(2,2)}$	2	-1

The decomposition matrix at $p = 2$ is

$$D_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

At $p = 3$, Lemma 3.4.2.1 says that the restrictions of χ_λ for $\lambda \neq (2,2)$ are all simple Brauer characters. The only nontrivial decomposition is $d\chi_{(2,2)} = d\chi_{(4)} + d\chi_{(1,1,1,1)}$. The decomposition matrix is

$$D_3 = \left(\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

where we ordered the ordinary simples in the order $(4), (1,1,1,1), (2,2), (3,1), (2,1,1)$.

3.4.3. Blocks.

Definition 3.4.3.1. Two simple Brauer characters ϕ, ϕ' are *linked* if there is an ordinary χ where both ϕ and ϕ' appear in $d\chi$. Let the *blocks* $Bl(G)$ of G be the set of equivalence classes of simple Brauer characters under the relation generated by $\phi \sim \phi'$ if ϕ and ϕ' are linked.

The blocks are exactly the blocks of rows if we attempt to minimally write the decomposition matrix in block diagonal form.

3.5. (Dec 10) End Times

3.5.1. Finishing the CDE triangle. Note that if A is a finite-dimensional k-algebra, then there is a pairing

$$K^0(A) \times K_0(A) \to \mathbf{Z}$$

which sends $P, V \mapsto \dim_k \operatorname{Hom}_A(P, V)$. Since P is projective, this is bilinear in short exact sequences (3.1.2.1 iv). Denote this pairing by $\langle [P], [V] \rangle = \dim_k \operatorname{Hom}_A(P, V)$. Lemma 3.1.2.4 implies that the classes of simples and projective covers of simples form dual bases under this pairing.

Lemma 3.5.1.1. *Let $x \in K_0(KG)$ and $y \in K^0(kG)$. Then*

$$\langle ey, x \rangle = \langle y, dx \rangle.$$

Proof. Assume $x = [P]$ and $y = [V]$. Let $\bar{V}$ be the reduction of a G-stable lattice in V and $\tilde{P}$ be a lift of P to RG. By Theorem 3.4.1.1,

$$\begin{aligned} \dim_k \operatorname{Hom}_{kG}(P, \bar{V}) &= \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_P(g^{-1}) \phi_{\bar{V}}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\tilde{P}}(g^{-1}) \chi_V(g) \\ &= \dim_K \operatorname{Hom}_{KG}(\tilde{P}, V), \end{aligned}$$

as desired. $\square$

Express the Cartan, decomposition, and lifting homomorphisms c, d, e as matrices C, D, E in terms of the bases of simples and indecomposable projectives. Theorem 3.3.1.4 says $C = DE$. Lemma 3.5.1.1 says $D = E^T$ (here we use that the simples and projectives are dual bases to say that the matrix of the adjoint is the transpose of the matrix). Thus $C = DD^T$. Thus, full knowledge of the Brauer and ordinary character tables lets you compute the Cartan matrix.

3.5.2. Blocks.

Definition 3.5.2.1. Two simple Brauer characters ϕ, ϕ' are *linked* if there is an ordinary χ where both ϕ and ϕ' appear in $d\chi$. Let the *blocks* $Bl(G)$ of G be the set of equivalence classes of simple Brauer characters under the relation generated by $\phi \sim \phi'$ if ϕ and ϕ' are linked.

By definition, the decomposition matrix is block diagonal with blocks indexed by... blocks. Say that an ordinary character χ belongs to a block if the constituents of $d\chi$ belong to that block. Note also that ϕ and ϕ' are linked if and only if $C_{\phi\phi'} \neq 0$.

Example 3.5.2.2. [Ser78, §16.4] If V is a simple KG-module with $\dim V$ divisible by the highest power of p dividing $|G|$, then V reduces to a simple projective kG-module. This is when a block has a single element.

Definition 3.5.2.3. If χ is an ordinary character of G, the associated *central character* is the function ω taking a conjugacy class to

$$\omega_\chi(C) = |C|\frac{\chi(C)}{\chi(1)}.$$

Lemma 3.5.2.4. *For any complex character χ, the associated character ω_χ is an algebraic integer.*

Proof. Let $x = \sum_{g \in C} g \in \mathbf{C}G$; then x is finite over $\mathbf{Z}G$. Hence its action on a $\mathbf{C}G$-module is by an algebraic integer. By definition $\operatorname{tr}(x; V_\chi) = |C|\chi(C)$, but x acts a scalar on V_χ, so x is acting by $\frac{|C|\chi(C)}{\chi(1)}$. $\square$

Proposition 3.5.2.5. *Let $(R, \mathfrak{m}, k, K)$ be a p-modular system. Two absolutely simple ordinary characters χ, χ' of KG are in the same block if and only if*

$$\omega_\chi \equiv \omega_{\chi'} \mod \mathfrak{m}.$$

Proposition 3.5.2.6. *Let $(R, \mathfrak{m}, k, K)$ be a p-modular system. For simple Brauer characters ϕ, ϕ' of G, the following are equivalent:*

i. ϕ and ϕ' are in the same block
ii. ϕ and ϕ' are in the same block of the Cartan matrix C;
iii. $\operatorname{Ext}^*(L_\phi, L_{\phi'}) \neq 0$*;*
iv. when $A = kG$ is broken into indecomposable two-sided ideals $A = Ae_1 \oplus \cdots \oplus Ae_r$, there is an i such that $e_i L_\phi \neq 0 \neq e_i L_{\phi'}$.

Proofs omitted. Let us point out that this means the blocks are intrinsic to characteristic p, even though they can be computed entirely using characteristic zero. This is all part of the fun :)

3.5.3. The symmetric group. How are the simple Σ_n-representations indexed over a field of positive characteristic?

Definition 3.5.3.1. Let p be a prime. A partition $\lambda \vdash n$ is *p-regular* if it does not have p parts of equal size.

Lemma 3.5.3.2. *The number of p-regular partitions is equal to the number of p-regular conjugacy classes of Σ_n.*

Proof. A class $\sigma \in \Sigma_n$ is p-regular if all cycle lengths are prime to p. Thus the generating function for the number of p-regular classes is

$$F_1(t) = \prod_{p \nmid i} \frac{1}{1-t^i}.$$

Now the generating function for the number of p-regular partitions is

$$F_2(t) = \prod_{i} \frac{1-t^{pi}}{1-t^i}.$$

But now

$$\frac{1}{1-t^i} = \frac{1-t^{pi}}{1-t^i}\frac{1-t^{p^2 i}}{1-t^{pi}}\cdots$$

so $F_1 = F_2$. ☐

Brauer's theorem tells us the p-regular partitions are a viable candidate for indexing the simple Σ_n-modules. It turns out they do index a natural construction: For $\lambda \vdash n$, let $M_\lambda = \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_n} k$. Define the *Specht module*

$$S^\lambda = \cap_{f: M_\lambda \to M_\mu, \mu > \lambda} \ker(f)$$

and

$$S^{\lambda\perp} = \sum_{f: M_\mu \to M_\lambda, \mu > \lambda} \mathrm{im}(f).$$

Lemma 3.5.3.3. *If k is of characteristic zero, then $M_\lambda = S^\lambda \bigoplus S^{\lambda\perp}$. If k has characteristic p,*

- *i. $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is zero or simple.*
- *ii. D^λ is nonzero if and only if λ is p-regular.*
- *iii. $\{D^\lambda\}_{\lambda p\text{-regular}}$ is a list of all isomorphism classes of simple Σ_n-modules.*

Example 3.5.3.4. When $\lambda = (n-1, 1)$, the Specht module S^λ is $\{x \in k^n \mid \sum_i x_i = 0\}$, and if $p \mid n$, then $S^{\lambda\perp} \cap S^\lambda$ is spanned by $(1, 1, \ldots, 1)$.

Question 3.5.3.5 (Open). What is a formula for the character of D^λ? What is a formula for the decomposition matrix of Σ_n?

There is a nice algorithm to decide whether two representations of Σ_n are in the same block.

Definition 3.5.3.6. $\lambda \vdash n$ is a *p-core* if λ has no p-rim hooks.

Proposition 3.5.3.7 ([JK81], 2.7.16). *For each partition λ, there is a unique p-core partition $\tilde{\lambda}$ such that removing p-rim hooks from λ until no p-rim hooks are left yields λ.*

Theorem 3.5.3.8 (Nakayama's conjecture). *Two $\lambda, \mu \vdash n$ correspond to the same block of $k\Sigma_n$ if and only if $\tilde{\lambda} = \tilde{\mu}$.*

3.5.4. Affine $\widehat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric groups. Recall that $\mathfrak{sl}_n$ is generated by the elements $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$ for $1 \leq i < n$. If we set $H_i = [E_i, F_i] = E_{i,i} - E_{i+1,i+1}$, then these satisfy the relations

$$[E_i, F_j] = \delta_{ij} H_i, \qquad [H_i, E_j] = c_{ij} E_j, \qquad [H_i, F_j] = -c_{ij} F_j,$$
$$(adE_i)^{1-c_{ij}} E_j = 0, (adF_i)^{1-c_{ij}}(F_j) = 0$$

where

$$c = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is the Cartan matrix of $\mathfrak{sl}_n$.

Definition 3.5.4.1. The Kac-Moody algebra $\hat{\mathfrak{sl}}_p$ has generators $E_i : 1 \leq i \leq p$ and the relations above with the Cartan matrix

$$\hat{c}_{ij} = \begin{cases} 2 & i \equiv j \mod p \\ -1 & i \equiv j \pm 1 \mod p \\ 0 & \text{else} \end{cases}.$$

Recall the Young-Jucys-Murphy elements $X_i = \sum_{j=1}^{i-1} (ji) \in \mathbf{Z}\Sigma_i$. The same proof as over $\mathbf{C}$ shows that the possible eigenvalues of X_i in a representation of Σ_i are integers in the interval $[-i, i]$. When k is of characteristic p, this means X_i has eigenvalues in $\mathbf{F}_p$.

Definition 3.5.4.2. Define

$$E_i : k\Sigma_n\text{–mod} \to k\Sigma_{n-1}\text{–mod}$$

by sending V to the generalized eigenspace $V[i]$ for X_n at $i \in \mathbf{F}_p$.

Since X_n commutes with Σ_{n-1}, we see $V[i]$ has an action of Σ_{n-1}, and

$$\bigoplus_{i \in \mathbf{F}_p} E_i = \mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}} .$$

By abstract nonsense, this implies that

$$\mathrm{Ind}^{\Sigma_n}_{\Sigma_{n-1}} = \bigoplus_{i \in \mathbf{F}_p} F_i$$

where $F_i : k\Sigma_n\text{–mod} \to k\Sigma_{n+1}\text{–mod}$ is a biadjoint functor to E_i.

These induce linear transformations on

$$\mathcal{R} = (\bigoplus_{n=1}^{\infty} K(k\Sigma_n)) \otimes \mathbf{C}.$$

The functors E_i and F_i induce linear transformations $E_i : \mathcal{R} \to \mathcal{R}$ and $F_i : \mathcal{R} \to \mathcal{R}$.

Theorem 3.5.4.3 (Grojnowski [Gro99]). *The functors E_i, F_i generate an action of $\widehat{\mathfrak{sl}_p}(\mathbf{C})$ on $\mathcal{R}$.*

Furthermore, the representation $\mathcal{R}$ of $\widehat{\mathfrak{sl}_p}$ is explicitly described: it is the *basic representation* generated by the highest weight vector $[1] \in K(k\Sigma_1)$. For more on this, see the survey article [BK03] and the original paper of Grojnowski [Gro99].

This is one entry into the story of *categorification:* can you make K_0 of one category into a representation of something else? In this case, $\coprod_n Rep(\Sigma_n)$ becomes a *categorical representation* of $\widehat{\mathfrak{sl}_p}$.

3.5.5. Outlook. In modular representation theory of GL_n (semisimple algebraic groups), the representations $\nabla_\lambda = H^0(G/B_-, \mathcal{O}(\lambda))$ are no longer simple. These play a similar role to characteristic zero representations in the modular theory for finite groups: in between the projectives and simples there is an easier class of "standard" modules.

Definition 3.5.5.1. $L_\lambda \subseteq H^0(G/B_-, \mathcal{O}(\lambda))$ is the simple subrepresentation generated by a highest weight vector.

Example 3.5.5.2. For GL_2 acting on $k[x,y]_p$, x^p generates a simple subrepresentation $k\{x^p, y^p\}$ of dimension 2.

The numbers $[L_\mu : V_\lambda]$ are analogous to the decomposition numbers. Define $\Delta_\lambda = \nabla^*_{w_0\lambda}$ to be the *Weyl module*

Conjecture 3.5.5.3 (Lusztig). Under certain assumptions on p and λ, $[L_\mu] = \sum_\lambda \pm P_{\lambda,\mu}(1)[\Delta_\lambda]$ where $P_{\lambda,\mu}$ are the *Kazhdan-Lusztig polynomials*.

The Kazhdan-Lusztig polynomials are certain polynomials defined by the geometry of the complex flag variety G/B. They also govern similar questions about the infinite-dimensional complex representation theory and the representation theory of quantum groups.

The Lusztig conjecture has since been established for very large p. See [CW21] for a discussion of this area.

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