A PROOF OF AN IDENTITY OF HOCHSCHILD

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Lemma. *Let* A *be a commutative algebra over* $\mathbb{F}_p$, $D \in \mathrm{Der}(A)$, $f \in A$. *Then*

$$D^{p-1}(f^{p-1}Df) = f^{p-1}D^p(f) - D(f)^p.$$

Proof. It suffices to prove the lemma for the polynomial ring $A = \mathbb{F}_p[x_0, x_1, \ldots]$ in countably many variables with $f = x_0$ and $\overline{D} \in \mathrm{Der}(A)$ the derivation defined by $\overline{D}(x_i) = x_{i+1}$ for all $i \geq 0$.

Now let $B = \mathbb{Z}/p^2\mathbb{Z}[x_0, x_1, \ldots]$ and let D denote the derivation defined by $D(x_i) = x_{i+1}$ for all $i \geq 0$. We have $B/pB = A$, and $D \equiv \overline{D} \mod p$. Lifting to B allows us to obtain the left-hand side of the desired identity in the following way:

$$D^p(f^p) = D^{p-1}(D(f^p)) = pD^{p-1}(f^{p-1}Df). \tag{1}$$

Now applying the Leibniz rule p times to $D^p(f^p)$, we obtain

$$D^p(f^p) = \sum_{i_1+\cdots+i_p=p} \frac{p!}{i_1!\cdots i_p!} D^{i_1}(f)\cdots D^{i_p}(f).$$

Given $(i_1, \ldots, i_p)$, let r_i be the number of $j \in \{1, \ldots, p\}$ with $i_j = i$. Since the derivatives of f commute, the term

$$\frac{p!}{i_1!\cdots i_p!} D^{i_1}(f)\cdots D^{i_p}(f)$$

appears $p!/(r_1!\cdots r_p!)$ times in the above sum. Hence, all terms in the sum disappear except for two cases: those where $i_j = p$ for some j, corresponding to the term $f^{p-1}D^p(f)$ which appears p times; or those all i_j are equal, corresponding to the term $p!D(f)^p$, appearing once. Thus,

$$D^p(f^p) = pf^{p-1}D^p(f) + p!D(f)^p. \tag{2}$$

Equating (1) and (2), then dividing by p, gives the desired formula. $\square$

Remark. This identity appeared in the Seminaire Claude Chevalley in 1958, in a talk by Seshadri on Cartier operators [4, Lemma 2], where it is attributed to Hochschild. This was in the same year as Cartier's paper [2, p. 203, (42)], where the identity appears in connection with logarithmic differential forms. Cartier cites [1, p. 59] for proof of this identity. The proof by Barsotti uses a similar technique as above, but is more complicated. That proof also appears in a Stack Exchange post [3].

References

[1] Iacopo Barsotti. Repartitions on abelian varieties. *Illinois Journal of Mathematics*, 2(1):43–70, 1958.

[2] Pierre Cartier. Questions de rationalité des diviseurs en géométrie algébrique. *Bulletin de la Société Mathématique de France*, 86:177–251, 1958.

[3] Ben (https://math.stackexchange.com/users/164207/ben). Conceptual explanation for the identity of Hochschild about derivations. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3038006 (version: 2018-12-13).

[4] Conjeerveram Srirangachari Seshadri. L'opération de Cartier. Applications. *Séminaire Claude Chevalley*, 4:1–26, 1958.

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