Regular holonomic $\mathcal{D}$-modules and equivariant Beilinson-Bernstein

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Abstract

These are notes for Kazhdan-Lusztig conjecture seminar, Winter 2020. Let G be an algebraic group, K a closed subgroup, and X a G-space with finitely many K-orbits. For example, we may take $K = B$ and X to be the flag variety. I will show how to localize $(\mathfrak{g}, K)$-modules to X à la Beilinson and Bernstein, and show that this localization is a regular holonomic $\mathcal{D}$-module. This lets us investigate $(\mathfrak{g}, K)$-modules via perverse sheaves on X, as will be discussed in the future. In the course of the talk, I will define every word in the title and give examples.

Throughout, we will work over $\mathbb{C}$. Given a morhpism of varieties $f : X \to Y$, then f_*, f^* will denote the non-derived pushforward and pullback of $\mathcal{D}$-modules, and f_+, f^+ the derived versions.

1 Holonomic $\mathcal{D}$-modules

Let X be a smooth variety over $\mathbb{C}$. Recall that $\mathcal{D}_X$ carries a PBW filtration $\{F_i^{PBW}\mathcal{D}_X\}_{i\geq 0}$ satisfying $\operatorname{gr}_{PBW}\mathcal{D}_X = \operatorname{Sym}\mathcal{T}_X$. For $\pi : T^*X \to X$, we have $\operatorname{Sym}\mathcal{T}_X = \pi_*\mathcal{O}_{T^*X}$. Taking associated graded of a compatibly filtered $\mathcal{D}$-module thus gives a module over $\mathcal{O}_{T^*X}$; while this module depends on the filtration, its support does not (given certain necessary conditions).

Definition 1.1. A *good filtration* on a $\mathcal{D}_X$-module $\mathcal{M}$ is an exhaustive nonnegative $\mathcal{O}_X$-module filtration $\{F_i\mathcal{M}\}_{i\geq 0}$ such that

1. it is compatible with the PBW filtration: $(F_i^{PBW}\mathcal{D}_X)F_j\mathcal{M} \subseteq F_{i+j}\mathcal{M}$;

2. the associated graded $\mathrm{gr}_F \mathcal{M}$ is coherent over $\mathrm{gr}_{PBW} \mathcal{D}_X$.

A $\mathcal{D}_X$-module with a good filtration is coherent. Conversely, every coherent $\mathcal{D}_X$-module has a good filtration: if X is affine and $S \subseteq \Gamma(X, \mathcal{M})$ is a finite generating set, then let $F_i\mathcal{M} = F_i^{PBW} S$. In general, globalize this construction.

Recall for a coherent sheaf of $\mathcal{O}_Y$-modules $\mathcal{F}$, the support of $\mathcal{F}$ is $\mathrm{supp}(\mathcal{F}) = \{x \in Y \mid \mathcal{F}_x \neq 0\}$, and the *characteristic cycle* of $\mathcal{F}$ is

$$CC(\mathcal{F}) = \sum_{Z \text{ components of } \mathrm{supp}(\mathcal{F})} m_Z[Z],$$

an algebraic cycle. The only facts we need about the multiplicities m_Z is that they are nonnegative, and that CC is additive on exact sequences.

Proposition 1.2 ([3], Proposition D.3.1). *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, the characterstic cycle of $\mathrm{gr}_F \mathcal{M}$ does not depend on the choice of a good filtration F.*

Proof. The proof is by Bernstein's trick [2, §8]. Say two filtrations F and F' are *neighboring* if for all i,

$$F_i\mathcal{M} \subseteq F'_i\mathcal{M} \subseteq F_{i+1}\mathcal{M}.$$

First, we show that if F and F' are neighboring, then $CC(\mathrm{gr}_F \mathcal{M}) = CC(\mathrm{gr}_{F'} \mathcal{M})$. The condition that F and F' are neighboring gives maps induced by inclusion

$$\varphi_i : F_i\mathcal{M}/F_{i-1}\mathcal{M} \to F'_i\mathcal{M}/F'_{i-1}\mathcal{M}.$$

$$0 \to \ker\varphi \to \mathrm{gr}_F \mathcal{M} \to \mathrm{gr}_{F'} \mathcal{M} \to \mathrm{coker}\,\varphi \to 0.$$

Then $\ker\varphi_i = (F'_{i-1}\mathcal{M} \cap F_i\mathcal{M})/F_{i-1}\mathcal{M} = F'_{i-1}\mathcal{M}/F_{i-1}\mathcal{M}$ since F and F' are adjacent. As $\mathrm{coker}\,\varphi_i = F'_i\mathcal{M}/F_i\mathcal{M}$, so $\mathrm{coker}\,\varphi \cong (\ker\varphi)[1]$. Additivity of the characteristic cycle shows the desired result.

Now given arbitrary good filtrations F and F' and an integer k, define filtrations $G^{(k)}$ by $G_i^{(k)}\mathcal{M} = F_i\mathcal{M} + F'_{i+k}\mathcal{M}$. Certainly $G^{(k)}$ and $G^{(k+1)}$ are adjacent for all k. Since F' is good, for $k << 0$ such that the generators of $\mathrm{gr}_{F'} \mathcal{M}$ lie in $F_{-k}\mathcal{M}$, $G^{(k)} = F$. Since F is good, for $k >> 0$ such that the generators of $\mathrm{gr}_F \mathcal{M}$ lie in $F'_k\mathcal{M}$, $G^{(k)} = F'[k]$. This proves the claim. □

Definition 1.3. Given a $\mathcal{D}_X$-module $\mathcal{M}$, let F be a good filtration. The singular support of $\mathcal{M}$ is

$$SS(\mathcal{M}) = \operatorname{supp} \pi^*(\operatorname{gr}_F \mathcal{M}) \subseteq T^*X.$$

The characteristic cycle is

$$CC(\mathcal{M}) = CC(\operatorname{gr}_F \mathcal{M}).$$

Example 1.4 ([3], Example 2.2.4)**.** Suppose $\mathcal{M}$ is a vector bundle of rank r with flat connection on X. Then the filtration $F_0\mathcal{M} = 0, F_1\mathcal{M} = \mathcal{M}$ has $gr_F\mathcal{M} = \mathcal{O}_X^r$. Hence, $CC(\mathcal{M}) = r[X]$, where $X \to T^*X$ via the zero section.

Proposition 1.5. *If* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *is a short exact sequence of* $\mathcal{D}_X$*-modules, then*

$$CC(\mathcal{M}) = CC(\mathcal{M}') + CC(\mathcal{M}'').$$

Proof. A good filtration on $\mathcal{M}$ defines compatible filtrations on $\mathcal{M}'$ and $\mathcal{M}''$, giving a short exact sequence of associated graded modules. □

The following is the first main theorem on singular support.

Proposition 1.6 (Bernstein's inequality)**.** *If* $\mathcal{M}$ *is a coherent* $\mathcal{D}_X$*-module and* Z *is an irreducible component of* $SS(\mathcal{M})$*, then* $\dim Z \geq \frac{1}{2} \dim T^*X$*.*

There is a refinement due to Gabber: the support is a coisotropic variety with respect to the Poisson structure of T^*X. There is an elementary proof for $X = \mathbb{A}^n$ due to Bernstein [1]. A general proof is in [3][Section 2.3], following [2]. The sketch of the argument is: if the support is not dense in the zero section of T^*X, restrict to a locally closed subvariety and induct (via Kashiwara's Lemma).

Definition 1.7. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is *holonomic* if $\mathcal{M} = 0$ or

$$\dim SS(\mathcal{M}) = \dim X = \frac{1}{2} \dim T^*X.$$

By Proposition 1.5, we have that holonomic $\mathcal{D}_X$-modules are closed under subquotients and extensions, that is, they form a thick subcategory of $\mathcal{D}_X$-modules. Further, as the characteristic cycle is additive and the dimension of the support can't decrease, holonomic $\mathcal{D}_X$-modules are of finite length.

Let $D_h^b(\mathcal{D}_X)$ denote the bounded derived category of $\mathcal{D}_X$-modules with holonomic cohomology. We will need the follwing statements:

Proposition 1.8 ([3], Theorem 3.2.3). *If $f : X \to Y$ is a morphism of smooth varieties, then $f_+ : D_h^b(X) \to D_h^b(Y)$ and $f^+ : D_h^b(Y) \to D_h^b(X)$.*

A more down-to-earth statement is: the higher pullbacks and push-forwards of holonomic $\mathcal{D}$-modules are holonomic.

Also, Bernstein claims this, but it's not obvious to me:

Proposition 1.9 ([2], Section 9). *If $\pi : X \to Y$ is a smooth morphism and $\pi^*\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module.*

2 Regular $\mathcal{D}$-modules

2.1 Regular meromorphic connections

Let C be a smooth curve over $\mathbb{C}$ with function field K. We first investigate connections over K, that is, connections on the generic point of C.

Definition 2.1. A *meromorphic connection* on C is a finite-dimensional K-vector space M equipped with $\nabla : M \to \Omega^1_{K/\mathbb{C}} \otimes_K M$ satisfying the Leibniz rule: $\nabla(fu) = df \otimes u + f\nabla u$.

The difference between meromorphic connections is a linear map, so by picking a basis $M \cong K^n$, any meromorphic connection is of the form $d + A$ for $A \in M_n(K)$.

Example 2.2. $C = \mathbb{P}^1$, $K = \mathbb{C}(x)$, $M = K$ with $\nabla = d - \alpha\frac{dx}{x}$.

Definition 2.3. A meromorphic connection M is *regular* at p if there exists a basis such that if $\nabla = d + A$ in this basis, then $ord_p A \geq -1$.

Proposition 2.4. *The following are equivalent for a meromorphic connection (M, ∇):*

- *M is regular at p;*
- *if x is a uniformizer for $\mathcal{O}_{C,p}$ and ∂ is a derivation of $K/\mathbb{C}$ with $\partial(x) = 1$, then for all $u \in M$, there exists an $\mathcal{O}_{C,p}$ lattice $u \in L \subseteq M$ stable under $x\nabla_\partial$;*
- *for x, ∂ as above, for all $u \in M$ there exists $f \in K[T]$ such that $f(x\nabla_\partial)u = 0$.*

It follows that the latter criteria hold independent of the choice of a uniformizer x for $\mathcal{O}_{C,p}$. It follows from the third criterion that regular meromorphic connections are closed under subquotients and extensions, that is, they form a thick subcategory of meromorphic connections.

Definition 2.5. A $\mathcal{D}_C$-module $\mathcal{M}$ is regular if and only if, given a completion $\overline{C}$, the associated meromorphic connection to $\mathcal{M}$ is regular at all $p \in \overline{C}$.

2.2 General varieties

Definition 2.6. Let X be a variety. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular holonomic if and only if for all morphisms $k : C \to X$ of curves into X and all $i \geq 0$, $H^i k^+(\mathcal{M})$ is a regular $\mathcal{D}_C$-module.

Regular $\mathcal{D}_C$-modules are closed under subquotients and extensions, so regular holonomic $\mathcal{D}_X$-modules are also closed under subquotients and extensions.

Proposition 2.7. *For $\pi : X \to Y$ a smooth morphism, $\mathcal{M}$ a holonomic $\mathcal{D}_Y$-module, $\mathcal{M}$ is regular holonomic if and only if $\pi^+\mathcal{M}$ is regular holonomic.*

Proof. This follows from curve-testing: if $k : C \to X$, then $k^+\pi^+\mathcal{M} = (\pi \circ k)^+\mathcal{M}$. Conversely, maps $k : C \to Y$ locally lift to X since π is formally smooth. □

Let $D^b_{rh}(\mathcal{D}_X)$ denote the bounded derived category of regular holonomic $\mathcal{D}_X$-modules.

Proposition 2.8 ([3], Theorem 6.1.5)**.** *If $f : X \to Y$, then $f^+ : D^b_{rh}(\mathcal{D}_Y) \to D^b_{rh}(\mathcal{D}_X)$ and $f_+ : D^b_{rh}(\mathcal{D}_X) \to D^b_{rh}(\mathcal{D}_Y)$.*

Proof of the statement for pullbacks. The statement for pullback follows from Propositions 1.8 and just as in 2.7. □

3 Equivariant $\mathcal{D}$-modules

Throughout this section, let G be an algebraic group acting on the variety X, $K \subseteq G$ a closed subgroup. The action gives a morphism $\sigma : G \times X \to X$ satisfying

$$\sigma \circ (1 \times \sigma) = \sigma \circ (m \times 1),$$

for $m : G \times G \to G$ the multiplication.

Definition 3.1. A $\mathcal{D}_X$-module $\mathcal{M}$ is *weakly G-equivariant* if there is an isomorphism of $\mathcal{O}_G \boxtimes \mathcal{D}_X$-modules $\varphi : \sigma^*\mathcal{M} \to \pi_2^*\mathcal{M}$ satisfying the cocycle condition: on $G \times G \times X$,

$$\begin{array}{ccccc} (1\times\sigma)^*\sigma^*\mathcal{M} & \xrightarrow{(1\times\sigma)^*\varphi} & (1\times\sigma)^*\pi_2^*\mathcal{M} & = & \pi_{23}^*\sigma^*\mathcal{M} \\ \Big\| & & & & \Big\downarrow{\scriptstyle \pi_{23}^*\varphi} \\ (m\times 1)^*\sigma^*\mathcal{M} & \xrightarrow{(m\times 1)^*\varphi} & (m\times 1)^*\pi_2^*\mathcal{M} & = & \pi_3^*\mathcal{M} \end{array} .$$

$\mathcal{M}$ is *equivariant* if φ is an isomorphism of $\mathcal{D}_{G\times X}$-modules.

Weak G-equivariance implies that global sections are a representation of G. If $\mathcal{M}$ is weakly equivariant, then there are two actions of $\mathfrak{g}$ on $\mathcal{M}$: one coming from differentiating the weakly equivariant structure, and the other from the map $\mathfrak{g} \to \mathcal{D}_X$ induced by differentiating the G-action on X.

Proposition 3.2 ([5])**.** *Let $\mathcal{M}$ be a weakly equivariant $\mathcal{D}_X$-module. Then $\mathcal{M}$ is equivariant if and only if the two actions of $\mathfrak{g}$ on $\mathcal{M}$ coincide.*

For instance, $\mathcal{D}_X$ is not equivariant, although it is weakly equivariant [5, 4]. The action of G on $\mathcal{D}_X$ is by conjugation of differential operators, so differentiating the action gives tha $\xi \in \mathfrak{g}$ acts on a local section D of $\mathcal{D}_X$ by $D \mapsto [\xi, D]$. However, $\mathfrak{g} \to \mathcal{D}_X$ acts on $\mathcal{D}_X$ by left-multiplication.

3.1 Equivariant localization

We now give a general construction for K-equivariant $\mathcal{D}_X$-modules, given a $(\mathfrak{g}, K)$-module, generalizing the localization construction in the Beilinson-Bernstein correspondence.

Definition 3.3. A $(\mathfrak{g}, K)$-module V is a $\mathfrak{g}$-module and an algebraic K-representation such that

- differentiating the K-action agrees with the action of $\mathfrak{k} \subseteq \mathfrak{g}$;
- $k(X \cdot v) = Ad(k)(X) \cdot kv$ for all $v \in V$, $k \in K$, $X \in \mathfrak{g}$.

Theorem 3.4. *Let V be a $(\mathfrak{g}, K)$-module. Define*

$$\mathrm{Loc}(V) = \mathcal{D}_X \otimes_{\underline{\mathcal{U}\mathfrak{g}}} \underline{V}.$$

Then $\mathrm{Loc}(V)$ is a K-equivariant $\mathcal{D}_X$-module.

This appears when $K = G$ in [4, Lemma 2.1].

Proof. Weak equivariance follows since $\mathcal{D}_X$ is weakly equivariant [4, Lemma 2.1], and constant sheaves are trivialized.

Once we know $\mathrm{Loc}(V)$ is weakly equivariant, to check that it is strongly equivariant, we just need to check that the $\mathfrak{k}$-actions induced by that of K via equivariance and by $\psi : \mathfrak{k} \to \mathcal{D}_X$ agree. The equivariant K-action is defined by $k \cdot (D \otimes v) = k \circ D \circ k^{-1} \otimes kv$. Differentiating, we obtain that for $\xi \in \mathfrak{k}$,

$$\xi \cdot (D \otimes v) = [\psi(\xi), D] \otimes v + D \otimes \xi v.$$

The action from $\psi : \mathfrak{k} \to \mathcal{D}_X$ is given by

$$\psi(\xi)D \otimes v = [\psi(\xi), D] \otimes v + D\psi(\xi) \otimes v = [\psi(\xi), D] \otimes v + D \otimes \xi v,$$

as desired. □

3.2 Regularity of certain equivariant $\mathcal{D}$-modules

We didn't have time during the talk to discuss this theorem.

Theorem 3.5 ([3], Theorem 11.6.1)**.** *Let Y be a smooth variety and K an algebraic group acting on Y with finitely many orbits. Then every coherent equivariant $\mathcal{D}_Y$-module is regular holonomic.*

Lemma 3.6 ([3], Proposition 1.7.1)**.** *If $X \supseteq U$ is an open subset such that $Z = X \setminus U$ is smooth, $j : U \to X$, $i : Z \to X$, then there is a distinguished triangle*

$$i_+ i^+ \mathcal{M} \to \mathcal{M} \to j_+ j^+ \mathcal{M} \to^{+1}$$

Proof of Theorem 3.5. Induct on the number of orbits. If Y is a single orbit, then $Y = K/K'$ for $K' \leq K$. Set $\sigma : K \times Y \to Y$, $\pi_2 : K \times Y \to Y$. If $\mathcal{M}$ is equivariant, then set $\pi : K \to Y$ the quotient map, $i : K \to K \times Y$ by $i(k) = (k^{-1}, kK')$.

$$\pi^* \mathcal{M} = (\pi_2 \circ i)^* \mathcal{M} = i^* \pi_2^* \mathcal{M} \cong i^* \sigma^* \mathcal{M}.$$

But, $\sigma \circ i(k) = K'$, so the latter pullback factors through the one-point space $\{K'\}$. Hence, $\pi^* \mathcal{M} \cong \mathcal{O}_K \otimes_{\mathbb{C}} M$ for M some vector space. Since $\mathcal{M}$ is coherent, M is finite-dimensional, so $\pi^* \mathcal{M}$ is regular holonomic. By Proposition 2.7, $\mathcal{M}$ is regular holonomic.

Inductive step: pick a closed orbit, use Lemma 3.6, use that regular holonomic modules are closed under pullback, pushforward, and extensions. □

3.3 Equivariant Beilinson-Bernstein

Now let G be a semisimple group, $B \subseteq G$ a Borel, $X = G/B$ the flag variety. Putting together all our constructions gives:

Theorem 3.7 (Equivariant Beilinson-Bernstein localization)**.** *The Beilinson-Bernstein equivalence restricts to an equivalence*

$$\mathrm{Loc} : (\mathfrak{g}, B)\text{-mod} \leftrightarrows \mathcal{D}_X\text{-mod}^B : \Gamma,$$

where $\mathcal{D}_X\text{-mod}^B$ denotes the category of B-equivariant $\mathcal{D}_X$-modules. Further, if $V \in (\mathfrak{g}, B)\text{-mod}$, then $\mathrm{Loc}(V)$ is regular holonomic.

References

[1] Joseph Bernstein. The analytic continuation of generalized functions with respect to a parameter. *Funktsional'nyi Analiz i ego Prilozheniya*, 6(4):26–40, 1972.

[2] Joseph Bernstein. Algebraic theory of $\mathcal{D}$-modules. *available at http://www. math. uchicago. edu/ mitya/langlands. html*, 1983. Lecture notes.

[3] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *$\mathcal{D}$-modules, perverse sheaves, and representation theory.* Springer, 2007.

[4] András C Lőrincz and Uli Walther. On categories of equivariant $\mathcal{D}$-modules. *Advances in Mathematics*, 351:429–478, 2019.

[5] Michel van den Bergh. Some generalities on G-equivariant quasi-coherent $\mathcal{O}_X$ and $\mathcal{D}_X$-modules. *Preprint Université Louis Pasteur, Strasbourg*, 1994.