Introduction to $\mathcal{D}$-modules II

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Abstract

These are notes for the Beillinson-Bernstein localization seminar, Fall 2019. This is the second of two talks in a series on $\mathcal{D}$-modules. This talk covers pullback and pushforward, Kashiwara's Lemma, singular support, and that $\mathbb{P}^n$ is a $\mathcal{D}$-affine variety.

1 Global differential operators

1.1 Global operators on $\mathbb{P}^1$

Let's compute $\Gamma(\mathcal{D}_{\mathbb{P}^1})$. We have the standard two affine coordinate charts $U_0 = \mathbb{P}^1 \setminus \infty$, $U_1 = \mathbb{P}^1 \setminus 0$, with coordinates z and z^{-1}. We need to compute the relation between the derivations ∂_z and $\partial_{z^{-1}}$ in $U_0 \cap U_1$. By the Leibniz rule,

$$\partial_{z^{-1}}(z) = -z^2$$

and thus $\partial_{z^{-1}}|_{U_0 \cap U_1} = -z^2 \partial_z|_{U_0 \cap U_1}$. This calculation reflects that $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}(2)$. We also conclude that the global vector fields are

$$\Gamma(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}) = \mathrm{span}_{\mathbb{C}}\{\partial_z, z\partial_z, z^2\partial_z\}.$$

This follows since a section over U_0, of the form

$$(c_0 + c_1 z + \cdots + c_n z^n)\partial_z = (c_0 + (z^{-1})^{-1} + \cdots + c_n (z^{-1})^{-n})(-(z^{-1})^2 \partial_{z^{-1}}),$$

extends over ∞ if and only if $c_k = 0$ for $k > 2$.

What is the Lie algebra structure of $\Gamma(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$ under the bracket? If we set $E = -\partial_z$, $H = -2z\partial_z$, $F = z^2\partial_z$ (as in [3, p.22, (9)]), then we can compute

$$[H, E] = [-2z\partial_z, -\partial_z] = -2\partial_z = 2E$$

and similarly $[H,F]=-2F, [E,F]=H$. So, the Lie algebra is $\mathfrak{sl}_2$. This is no accident: $\mathbb{P}^1$ is the flag variety of $\mathfrak{sl}_2$. This induces a map $\varphi : \mathcal{U}\mathfrak{sl}_2 \to \Gamma(\mathcal{D}_{\mathbb{P}^1})$.

Let $\Omega = \frac{1}{2}H^2 + H + 2FE$ be the Casimir element.

Theorem 1.1. *The map* $\varphi : \mathcal{U}\mathfrak{sl}_2/(\Omega) \to \Gamma(\mathcal{D}_{\mathbb{P}^1})$ *is an isomorphism.*

Proof. First, we check the map is well-defined. The Casimir element $\Omega = \frac{1}{2}H^2 + H + 2FE$ maps to zero:

$$\varphi(\frac{1}{2}H^2) = 2z\partial_z z\partial_z = 2z^2\partial_z^2 + 2z\partial_z = \varphi(-2FE - H).$$

For $F_i\mathcal{D}_{\mathbb{P}^1}$ the usual degree filtration on $\mathcal{D}_{\mathbb{P}^1}$, we have

$$0 \to F_{i-1}\mathcal{D}_{\mathbb{P}^1} \to F_i\mathcal{D}_{\mathbb{P}^1} \to \operatorname{Sym}^i \mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}(2i) \to 0. \qquad (1)$$

This is because tensor powers and symmetric powers coincide for line bundles. Since the products of global vector fields surject onto $\operatorname{Sym}^i \mathcal{T}_{\mathbb{P}^1}$, we conclude that $\Gamma\mathcal{D}_{\mathbb{P}^1}$ is generated in degree one. Hence, φ is surjective.

To prove injectivity, we do a dimension count. On the left-hand side, we have the image of the PBW filtration; on the right-hand side, we have the degree filtration. The map φ preserves the filtration, so we may pass to associated graded.

Now $\operatorname{gr}^i \mathcal{U}\mathfrak{sl}_2 \cong \operatorname{Sym}^i \mathfrak{sl}_2 = \binom{i+2}{2}$, and as the ideal generated by the Casimir Ω is degree 2, we get $\dim F_i\mathcal{U}\mathfrak{sl}_2/(\Omega) = \dim \operatorname{gr}^i \mathcal{U}\mathfrak{sl}_2 + \dim \operatorname{gr}^{i-1} \mathcal{U}\mathfrak{sl}_2$. Hence,

$$\dim \operatorname{gr}^i \mathcal{U}\mathfrak{sl}_2/(\Omega) = \dim \operatorname{gr}^i \mathcal{U}\mathfrak{sl}_2 - \dim \operatorname{gr}^{i-2} \mathcal{U}\mathfrak{sl}_2 = \binom{i+2}{2} - \binom{i}{2} = 2i+1.$$

Now we count dimensions on the other side. Taking cohomology of our short exact sequence in (1) gives

$$\begin{aligned} 0 &\to \Gamma(F_{i-1}\mathcal{D}_{\mathbb{P}^1}) \to \Gamma(F_i\mathcal{D}_{\mathbb{P}^1}) \to \Gamma(\mathcal{O}(2i)) \\ &\to H^1(F_{i-1}\mathcal{D}_{\mathbb{P}^1}) \to H^1(F_i\mathcal{D}_{\mathbb{P}^1}) \to H^1(\mathcal{O}(2i)) \to 0. \end{aligned}$$

Now $H^1(\mathcal{O}(2i)) = 0$ for $i \geq 0$ and $H^1(F_0) = H^1(\mathcal{O}) = 0$, so we get $H^1(F_i) = 0$ for all $i \geq 0$, and

$$0 \to \Gamma F_{i-1} \to \Gamma F_i \to \Gamma\mathcal{O}(2i) \to 0.$$

In particular, the dimension of $\Gamma F_i/\Gamma F_{i-1}$ is $2i+1$. So $\operatorname{gr}^i \varphi$ is a surjective map between vector spaces of dimension $2i+1$. □

This theorem is the first case of the Beillinson-Bernstein correspondence. The theorem has two parts: it describes the global differential operators on the flag variety, and it states $\mathcal{D}$-modules are the same as modules over global differential operators.

Remark 1.2. Describing these global sections in coordinates is complicated: for instance

$$(z^2\partial_z)^2 = z^4\partial_z^2 - 2z^3\partial_z,$$

but neither of the latter are global differential operators. In particular, the formula at the bottom of p. 22 in [3] is incorrect.

1.2 $\mathcal{D}$-affine varieties

What is the difference between $\Gamma(\mathcal{D}_{\mathbb{P}^1})$ and $\mathcal{D}_{\mathbb{P}^1}$? For a smooth variety X over $\mathbb{C}$, we have an adjunction

$$\mathrm{Loc} : \Gamma(\mathcal{D}_X)\text{-mod} \rightleftarrows \mathcal{D}_X\text{-mod} : \Gamma$$

where $\mathrm{Loc}\, \mathcal{D}_X \otimes_{\Gamma(\mathcal{D}_X)} -$.

Definition 1.3. The variety X is *$\mathcal{D}$-affine* if the functors Γ and Loc are equivalences.

Proposition 1.4 (Abstract nonsense, [2] Proposition 1.4.4)**.** *X is $\mathcal{D}$-affine if and only if $\Gamma : \mathcal{D}_X\text{-mod} \to \Gamma(\mathcal{D}_X)\text{-mod}$ is exact and $\Gamma(\mathcal{M}) = 0$ implies $\mathcal{M} = 0$.*

The last condition is equivalent to $\mathcal{D}_X$-modules being *globally generated*. For a variety X, Serre's criterion states

$$\Gamma : Qcoh(X) \to \Gamma(\mathcal{O}_X)\text{-mod}$$

is an equivalence if and only if X is affine [1, Theorem III.3.7]. As the global sections functor coincides with global sections on $Qcoh(X)$, we conclude that if X is affine, then X is $\mathcal{D}$-affine. However, there are more $\mathcal{D}$-affine varieties. The goal of the rest of the talk is to prove:

Theorem 1.5. *$\mathbb{P}^n$ is a $\mathcal{D}$-affine variety.*

Remark 1.6. $\mathbb{P}^n$ is a *partial flag variety*, as SL_{n+1} acts on it with parabolic stabilizer $GL_n \rtimes \mathbb{G}_a^n$.

2 Pullback and pushforward

To prove more about $\mathcal{D}_X$-modules, we need more technology.

Definition 2.1. Let $\phi : X \to Y$ be a smooth morphism over S. Then for $\mathcal{M}$ a left $\mathcal{D}_Y$-module, the pullback $\phi^*\mathcal{M}$ is given by the usual pullback of $\mathcal{O}$-modules:

$$\phi^*\mathcal{M} = \mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{M}$$

as an $\mathcal{O}_X$-module. The action of $\mathcal{D}_X$ defined by the derivative of ϕ:

$$T\phi : T_X \to \phi^* T_Y$$

and $\xi \in T_X$ acts by the Leibniz rule on $\mathcal{O}_X \otimes \phi^{-1}\mathcal{M}$:

$$\xi(f \otimes m) = \xi(f) \otimes m + fT\phi(\xi)(m).$$

Pushforwards are more complicated. Following [3], observe that $\phi^{-1}\mathcal{M}$ is a $\phi^{-1}\mathcal{D}_Y$-module, so if we set $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\phi_{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{D}_Y$, then $\mathcal{D}_{X\to Y}$ is a $(\mathcal{D}_X, \phi^{-1}\mathcal{D}_Y)$-bimodule, which is exactly the gadget we need to turn a left $\phi^{-1}\mathcal{D}_Y$-module into a left $\mathcal{D}_X$-module.

This gadget lets us turn *right* $\mathcal{D}_X$-modules into *right* $\phi^{-1}\mathcal{D}_Y$-bimodules, so for $\mathcal{P}$ a right $\mathcal{D}_X$-module, we can define the pushforward

$$\int_\phi^0 \mathcal{P} = \phi.(\mathcal{P} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

To define the pushforward for left $\mathcal{D}$-modules, we have to use side-changing: $\mathcal{D}_{Y\leftarrow X}$ is a $(\phi^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule obtained from $\mathcal{D}_{X\to Y}$.

Definition 2.2. For $\mathcal{M}$ a left $\mathcal{D}_X$-module and $\phi : X \to Y$, the pushforward is given by

$$\int_\phi^0 \mathcal{M} = \phi.(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}).$$

The *derived pushforward* is given by

$$\int_\phi \mathcal{M} = R\phi.(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}),$$

and $\int_\phi^i \mathcal{M} = H^i(\int_\phi \mathcal{M})$.

There are issues with this derived definition: how does one compose left and right derived functors? I can't resolve them here. We need that there is a derived pushforward, and consequently we get a long exact sequence in cohomology. See [2, Section 1.5].

Example 2.3. Say $\varphi : \mathbb{A}^k \to \mathbb{A}^n$ is the inclusion of the space $\{x_{k+1} = \cdots = x_n = 0\}$ via the standard coordinates x_i. Then $\mathcal{D}_{X\to Y} = \mathbb{C}[x_1, \dots, x_k]\langle \partial_1, \dots, \partial_n\rangle = \mathcal{D}_X\langle \partial_{k+1} \dots, \partial_n\rangle$, The side-change is

$$\mathcal{D}_{Y\leftarrow X} = \langle \partial_{k+1}, \dots, \partial_n\rangle \mathcal{D}_X.$$

Hence for M a left $\mathcal{D}_{\mathbb{A}^k}$-module, $\int_\varphi^0 M = \langle \partial_{k+1}, \dots, \partial_n\rangle M$. (describe $\mathcal{D}_{\mathbb{A}^n}$ action.)

Example 2.4. Say $U \to Y$ is an open immersion. Then it induces an isomorphism on tangent spaces, so $\mathcal{D}_{X\to Y} = \mathcal{D}_U$. Hence, the pushforward $\int_\phi^0 \mathcal{M}$ coincides with the pushforward as $\mathcal{O}$-modules.

Let $i : Y \to X$ be a closed subvariety. Kashiwara's Lemma describes the $\mathcal{D}_X$-modules which are supported as $\mathcal{O}_X$-modules on Y, written $\mathcal{D}_X\text{-mod}_Y$.

Lemma 2.5 (Kashiwara, [2] Theorem 1.6.1)**.** *The pushforward* $\int_i^0 : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod}_Y$ *is an equivalence of categories.*

Example 2.6. If $j : p \to X$ is the inclusion of a point, then a $\mathcal{D}_p$-module is just a vector space. Let $\{x_i, \partial_i\}$ be local coordinates at x. Then locally at p, $\int_j^0 1 = \mathbb{C}[\partial_1, \dots \partial_n]\delta$ where $x_i\delta = 0$ for all i. This is the *Dirac delta function at* p, as a $\mathcal{D}$-module.

3 $\mathbb{P}^n$ is $\mathcal{D}$-affine

$\mathbb{P}^n$ is a $\mathbb{C}^*$-quotient; we will use this structure. The Lie algebra of $\mathbb{C}^\times = \operatorname{Spec}\mathbb{C}[t, t^{-1}]$ is spanned by $t\partial_t$, as it is left invariant.

Let V be a vector space with coordinate functions $\{x_0, \dots, x_n\}$ and corresponding vector fields $\{\partial_0, \dots, \partial_n\}$.

Proposition 3.1. *The infinitesimal action* $Lie(\mathbb{C}^\times) \to \mathcal{T}_V$ *of* $\mathbb{C}^\times$ *on* V *is given by*

$$t\partial_t \mapsto \xi = \sum_{i=0}^n x_i\partial_i.$$

The vector field $\xi = \sum_{i=0}^n x_i \partial_i$ is known as the *Euler vector field.* If f is a homogeneous polynomial in the variables x_i, then $\xi(f) = (\deg f) f$. The following lemma is essentially a consequence of this:

Lemma 3.2 ([3], Proposition 2.46)**.** *Let $\pi : V \setminus 0 \to \mathbb{P}(V)$ denote the projection. If $\mathcal{M}$ is a $\mathcal{D}_{\mathbb{P}(V)}$-module, then the $\mathbb{C}^\times$ grading on $\Gamma(\pi^*\mathcal{M})$ is the eigendecomposition of $\Gamma(\pi^*\mathcal{M})$ with respect to ξ.*

Theorem 3.3. *$\mathbb{P}(V)$ is $\mathcal{D}$-affine.*

Proof, following [3], Proposition 2.48, or [2], Theorem 1.6.5. First, we prove exactness. Let $\pi : V \setminus 0 \to \mathbb{P}(V)$ be the projection, $j : V \setminus 0 \to V$ the inclusion. The global sections functor on $\mathcal{D}_{\mathbb{P}(V)}$-mod factors as follows:

$$\begin{array}{ccc} \mathcal{D}_{V\setminus 0}\text{-mod} & \xrightarrow{\Gamma} & \Gamma(\mathcal{D}_{V\setminus 0})\text{-mod} \\ {\scriptstyle \pi^*}\big\uparrow & & \big\downarrow {\scriptstyle \ker_{\mathbb{C}^\times} = \ker(\xi)} \\ \mathcal{D}_{\mathbb{P}(V)}\text{-mod} & \xrightarrow{\Gamma} & \Gamma(\mathcal{D}_{\mathbb{P}(V)})\text{-mod} \end{array}$$

The pullback π^* is exact since π is smooth. Further, as $\Gamma(\mathcal{D}_{V\setminus 0}) = \Gamma(\mathcal{D}_V)$, and V is affine, the global sections functor coincides with the pushfoward $\int_j^0$. So, given $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ a short exact sequence of $\mathcal{D}_{\mathbb{P}(V)}$-modules, we get a long exact sequence

$$0 \to j_*(\pi^*\mathcal{M}') \to j_*(\pi^*\mathcal{M}) \to j_*(\pi^*\mathcal{M}'') \to \int_j^1 \pi^*\mathcal{M}' \to \cdots \ldots$$

Now the pushforward j_* is exact on $V \setminus 0$ (it's an isomorphism there) so as an $\mathcal{O}$-module, $\int_j^1 \pi^*\mathcal{M}'$ is supported on $\{0\}$. Hence, it is a direct sum of Dirac delta functions. As $\ker(\xi)$ is an exact functor, it suffices to show that the kernel of ξ on $\int_j^1 \pi^*\mathcal{M}'$ is zero. But if δ is a Dirac delta at zero, then letting $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, we compute

$$\left(\sum_{i=0}^n x_i\partial_i\right)\partial^\alpha\delta = \sum_{i=0}^n [x_i, \partial_i\partial^\alpha]\delta = \sum_{i=0}^n (-1-\alpha_i)\partial^\alpha\delta$$

and thus ξ acts on $\mathbb{C}[\partial_0, \ldots, \partial_n]\delta$ with strictly negative eigenvalues.

Now suppose that $\Gamma(\mathcal{M}) = 0$. Consider the grading on $\Gamma(\pi^*\mathcal{M}) = \bigoplus_d \Gamma(\pi^*\mathcal{M})_d$. If $\mathcal{M} \neq 0$, then there is some d so that $\Gamma(\mathcal{O}(d) \otimes \mathcal{M}) \neq 0$ [1, Theorem III.5.17], so $\Gamma(\pi^*\mathcal{M}) \neq 0$. It suffices to show that $\Gamma(\pi^*\mathcal{M}) = 0$, given $\Gamma(\pi^*\mathcal{M})_0 = 0$. The action of ∂_i lowers the grading by one, and the action of x_i lowers the grading by one. So

$\xi : \Gamma(\pi^*\mathcal{M})_d \to \Gamma(\pi^*\mathcal{M})_{d-1} \to \Gamma(\pi^*\mathcal{M})_d$. If $\Gamma(\pi^*\mathcal{M})_0 = 0$, then by induction, $\Gamma(\pi^*\mathcal{M})_d = 0$ for $d \geq 0$.

Now $x_i\Gamma(\pi^*\mathcal{M})_{-1} = 0$ for all i, so $\Gamma(\pi^*\mathcal{M})_{-1}$ is supported at $\{0\}$. But $\pi^*\mathcal{M}$ is on $V \setminus 0$, so $\Gamma(\pi^*\mathcal{M})_{-1} = 0$. Inductively repeating this argument gives $\Gamma(\pi^*\mathcal{M})_d = 0$ for $d < 0$. □

References

[1] Robin Hartshorne. *Algebraic geometry.* Springer, 1977.

[2] Ryoshi Hotta and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory.* Springer, 2007.

[3] Yi Sun. $\mathcal{D}$-modules and representation theory.