Introduction to $\mathcal{D}$-modules I

Joshua Mundinger

October 22, 2019

Abstract

These are notes for an introductory talk on $\mathcal{D}_X$-modules in the student Beilinson-Bernstein localization seminar, Fall 2019. The main references for this subject are [1, 2]. This lecture covers the definition of differential operators, left and right modules, and side-changing.

1 Definition of $\mathcal{D}_{X/S}$

1.1 Vector fields in algebraic geometry

Here we give an algebraic description of vector fields, suitable for algebraic geometry.

Definition 1.1. Given a morphism of commutative rings with unity $B \to A$, a *derivation* of A over B is a map $D : A \to A$ which is B-linear and satisfies the Leibniz rule

$$D(fg) = fD(g) + D(f)g.$$

The space of derivations of A over B is denoted $\mathrm{Der}_B(A)$.

For $D : A \to A$ satisfying the Leibniz rule, B-linearity is equivalent to $D(b) = 0$ for all $b \in B$. $\mathrm{Der}_B(A)$ is an A-module.

Proposition 1.2. *Let $X \to *$ be a smooth manifold. A derivation of $C^\infty(X)$ over $\mathbb{R}$ is the same as a vector field on X.*

Proof. Vector fields act as derivations on $C^\infty(X)$. Conversely, if D is a derivation of $C^\infty(X)$, then using partitions of unity, D defines a derivation on $C^\infty(X)$ for all $U \subseteq X$: if $f \in C^\infty(U)$, $p \in U$, take

$\varphi \in C_c^\infty(U)$ which is identically 1 in a neighborhood of p, and define $Df(p) = D(\varphi f)(p)$. It is an exercise to show this is well-defined.

Now suppose $U \subseteq X$ is a coordinate patch and $\mathbf{x} : U \to \mathbb{R}^n$ are local coordinates. If $f \in C^\infty(U)$ and $q \in U$ with $\mathbf{x}(q) = (t_1, \ldots, t_n)$, we have

$$f = f(q) + \sum_i (x_i - t_i) f_i$$

for smooth functions f_i, defined possibly in a possibly smaller neighborhood of q [3, Lemma 2.1]. Then for a derivation D of $C^\infty(X)$, $Df(q) = \sum_i d(x_i)(q) f_i(q)$. But $f_i(q) = \frac{\partial f}{\partial x_i}(q)$, so

$$Df(q) = \sum_i d(x_i)(q) \frac{\partial f}{\partial x_i}(q).$$

As q was arbitrary, we obtain $D = \sum_i d(x_i) \frac{\partial}{\partial x_i}$ on U. This shows that D is a vector field on U. Then glue patches. □

In this proof, partitions of unity showed that a derivation of global functions descended to compatible derivations of local functions. In a more general geometric context, we will have to use sheaves of derivations instead.

Let $\pi : X \to S$ be a morphism of ringed spaces. Here are the two key examples for this talk: first, if X is a smooth manifold, S is a point, and both are equipped with their rings of smooth real functions; second, if $X \to S$ is a morphism of schemes.

Definition 1.3. The *relative tangent sheaf* of X/S is the sheaf

$$\mathcal{T}(X/S) = \mathscr{D}er_{\pi^{-1}\mathcal{O}_S}(\mathcal{O}_X)$$

of $\pi^{-1}\mathcal{O}_S$-linear derivations $\mathcal{O}_X \to \mathcal{O}_X$.

Example 1.4. If $M \to M'$ is a submersion of manifolds, then the relative tangents $\mathcal{T}_{M/M'}$ are given by

$$0 \to \mathcal{T}_{M/M'} \to \mathcal{T}_{M/\mathbb{R}} \to \mathcal{T}_{M'/\mathbb{R}} \to 0,$$

which is dual to the Euler exact sequence.

1.2 The ring of differential operators

There are two definitions of the ring of differential operators. In general, they are not equal, but they agree in the case of a smooth morphism of schemes in characteristic zero.

Definition 1.5. $\mathcal{D}_{X/S}$ is the sub-sheaf of rings of $\mathscr{E}nd_{\pi^{-1}\mathcal{O}_S}(\mathcal{O}_X)$ locally generated by $\pi^{-1}\mathcal{O}_S$ and $\mathcal{T}_{X/S}$.

Definition 1.6. Let $B \to A$ be a morphism of commutative rings. The *Grothendieck differential operators of order* n, written $F_nD_{A/B}$ are exactly the B-linear functions $d : A \to A$ satisfying that for all $f \in A$,

$$[d, f] \in F_{n-1}D_{A/B},$$

where we view f as acting as multiplication on A, with $F_{-1}D_{A/B} = 0$.

The *ring of Grothendieck differential operators* of A/B is $D_{A/B} = \cup_n D_{A/B}$.

Let's calculate the Grothendieck differential operators of order 0 and 1. The order 0 differential operators are exactly the A-linear functions $A \to A$, so $F_0D_{A/B} = A$.

If $d \in F_1D_{A/B}$ then for all $f, g \in A$,

$$d(fg) = fd(g) + [d, f]g.$$

In particular,taking $g = 1$ gives

$$d(f) = fd(1) + [d, f].$$

The function d is a derivation if and only if $d(1) = 0$, and d is in $\mathcal{O}_X$ if and only if $[d, f] = 0$ for all f. This shows $F_1D_{A/B} = A \oplus \mathrm{Der}_B(A)$, via the projections $d \mapsto d(1)$ and $d \mapsto d - d(1)$.

Exercise 1.7. *$D_{A/B}$ is an almost commutative filtered ring: we have for $d \in F_nD_{A/B}$ and $d' \in F_mD_{A/B}$ that $d \circ d' \in F_{n+m}D_{A/B}$ and $[d, d'] \in F_{n+m-1}D_{A/B}$. Conclude*

$$\mathrm{gr}_F\, D_{A/B} = \oplus_{n\geq 0}F_nD_{A/B}/F_{n-1}D_{A/B}$$

is a commutative A-algebra.

Proposition 1.8. *[1, Exercise 2.1.16] If B contains a field of characteristic zero, $B \to A$ is of finite type, and $Der_B(A)$ is a locally free A-module, then $\mathrm{gr}\, D_{A/B} = \mathrm{Sym}^*_A \mathrm{Der}_B(A)$, induced by the map* $\mathrm{Der}_B(A) \cong \mathrm{gr}^1 D_{A/B}$.

Now one can define Grothendieck differential operators of $X \to S$ as the sheafification of local Grothendieck differential operators. The proposition above ensures that locally, Grothendieck differential operators are generated by vector fields.

Corollary 1.9. *If $X \to S$ is a smooth morphism of schemes in characteristic zero, our two definitions of differential opeartors agree, and $\mathcal{D}_{X/S}$ is generated by $\mathcal{T}_{X/S}$ subject to the relations $[\xi, f] = \xi(f)$ for $\xi \in \mathcal{T}_{X/S}$, $f \in \mathcal{O}_X$ and $\xi_1\xi_2 - \xi_2\xi_1 = [\xi_1, \xi_2]$ for $[-,-]$ the bracket on $\mathcal{T}_{X/S}$ and $\xi_i \in \mathcal{T}_{X/S}$.*

That these relations suffice can be checked by passing to associated graded.

Remark 1.10. *The filtration F corresponds to the PBW filtration on $\mathcal{D}_{X/S}$: $F_n\mathcal{D}_{X/S}$ is the span of morphisms whith are products of at most n vector fields. Then, a PBW theorem may be proven directly, as in [1, Corollary 2.1.8].*

Example 1.11. If $X = \mathbb{A}^n_k \to S = \operatorname{Spec} k$, (suppressing S from now on,) then $\Gamma(X, \mathcal{T}_{X/k}) = k[x_1, \ldots, x_n]\{\partial_1, \ldots, \partial_n\}$ with bracket $[\partial_i, \partial_j] = 0$ for all i and j and $\partial_i(x_j) = \delta_{ij}$. Hence

$$D_{X/k} = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

modulo the relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$. This is known as the Weyl algebra A_n.

2 $\mathcal{D}_X$-modules

From this point, our spaces are smooth varieties over $S = \operatorname{Spec} \mathbb{C}$. The base scheme S will be suppressed from the notation.

Definition 2.1. A $\mathcal{D}_X$-module is a sheaf on X which is a module over $\mathcal{D}_X$ and quasicoherent as an $\mathcal{O}_X$-module (with respect to $\mathcal{O}_X \subseteq \mathcal{D}_X$).

We have a presentation of $\mathcal{D}_X$, generated by $\mathcal{T}_X$ with certain relations. So, to define a $\mathcal{D}_X$-module structure on an $\mathcal{O}_X$-module is to specify the action of $\mathcal{T}_X$, satisfying certain relations. For a left $\mathcal{D}_X$-module, these relations coincide with those of a *connection.*

Proposition 2.2. *[2, Lemma 1.2.1] A* left *$\mathcal{D}_X$-module is the same as a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with a connection $\nabla : \mathcal{T}_X \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}$ satisfying*

1. $\nabla_{f\theta}(m) = f\nabla_\theta m$
2. $\nabla_\theta(fm) = f\nabla_\theta m + \theta(f)m$,
3. $[\nabla_{\theta_1}, \nabla_{\theta_2}] = \nabla_{[\theta_1,\theta_2]}$.

The last condition is that ∇ has zero curvature, and so is sometimes called *flatness*; it is also known as *integrability*.

Example 2.3. The ring of functions $\mathcal{O}_X$ with natural action by derivations of $\mathcal{T}_X$ makes it a $\mathcal{D}_X$-module.

2.1 Aside on smooth manifolds

For M a smooth manifold over $\mathbb{R}$, we have bijections

$$\begin{gathered}\{\text{rank } n \text{ vector bundles with flat connection}\}\\ \updownarrow\\ \{\text{rank } n \text{ locally constant sheaves}\}\\ \updownarrow\\ \{n\text{-dimensional representation of } \pi_1(M)\}.\end{gathered}$$

The first arrow sends (E, ∇) to $\{s \mid \nabla s = 0\}$. The integrability condition, with the Frobenius integrability theorem, implies that the result is a rank n locally constant sheaf. The second is standard algebraic topology: a locally constant sheaf is determined by its monodromy representation.

This is a version of the *Riemann-Hilbert correspondence.* A version also exists in the algebraic category, but I can't talk about it.

2.2 Right $\mathcal{D}_X$-modules

Proposition 2.4. *[2, Lemma 1.2.5] A* right *$\mathcal{D}_X$-module is the same as an $\mathcal{O}_X$-module $\mathcal{P}$ and a map $\nabla' : \mathcal{T}_X \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}$ satisfying*

1. $\nabla'_{f\theta}(m) = \nabla'_\theta(fm)$,
2. $\nabla'_\theta(fm) = f\nabla'_\theta m + \theta(f)m$,
3. $[\nabla'_{\theta_1}, \nabla'_{\theta_2}] = \nabla'_{[\theta_1,\theta_2]}$.

The right structure is given by: for $a \in \mathcal{P}$ and $\theta \in \mathcal{T}_X$, $a\theta = -\nabla'_\theta(a)$.

Example 2.5. The canonical bundle $\omega_X = \bigwedge^{\dim X} \Omega^1_X$ is a right $\mathcal{D}_X$-module under $\nabla'_\theta = \mathrm{Lie}_\theta$.

The philosophy is that left $\mathcal{D}$-modules are functions, and right $\mathcal{D}$-modules are measures. This will come up when defining the six functors for $\mathcal{D}_X$-modules.

2.3 Side-changing

The Weyl algebra has an anti-automorphism $x_i \mapsto x_i, \partial_i \mapsto -\partial_i$. Hence $A \cong A^{op}$, so a left-module may be considered as a right-module via the above. This works on a general affine variety, but not globally. Instead, we need a global twist.

The canonical bundle $\omega_{X/S}$ changes between left and right $\mathcal{D}$-modules. In local coordinates, for $a = f dx_1 \wedge \cdots dx_n$,

$$\mathrm{Lie}_{\partial_j} f dx_1 \wedge \cdots \wedge dx_n = (\partial_j f) dx_1 \wedge \cdots \wedge dx_n$$

and so in general

$$a\partial_j = -\partial_j f dx_1 \wedge \cdots \wedge dx_n,$$

suggesting the anti-automorphism of the Weyl algebra we discussed earlier. See [2, Lemma 1.2.6] for more details.

2.3.1 Hom and tensor

Tensoring two left $\mathcal{D}_X$-modules produces a left module. Tensoring a left $\mathcal{D}_X$ module $\mathcal{M}$ and a right $\mathcal{D}_X$-module $\mathcal{N}'$ gives a *right* $\mathcal{D}_X$-module, with

$$(m \otimes n)\theta = m \otimes n\theta - \theta m \otimes n.$$

Also, if $\mathcal{M}'$ and $\mathcal{N}'$ are right $\mathcal{D}_X$-modules, then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}')$ is a *left* $\mathcal{D}_X$-module with $(\theta\phi)(m) = \phi(m\theta) - \theta(\phi(m))$ [2, Proposition 1.2.9]. Then

$$\omega_X \otimes - : \mathcal{D}_X\text{-mod} \longleftrightarrow \mathcal{D}_X^{op}\text{-mod} : \mathrm{Hom}_{\mathcal{O}_X}(\omega_X, -)$$

is an equivalence between left- and right- $\mathcal{D}_X$-modules, since they are quasi-inverses on the underlying $\mathcal{O}_X$-modules as ω_X is a line bundle.

References

[1] Victor Ginzburg. Lectures on $\mathcal{D}$-modules, 1998.

[2] Ryoshi Hotta and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory.* Springer, 2007.

[3] John Milnor. *Morse Theory.* Princeton University Press, 1963.