A Module-Theoretic Approach to Matroids

Joshua Mundinger

Swarthmore College

March 17, 2018

Joint work with Colin Crowley and Noah Giansiracusa

What is a tropical linear space?

David Speyer defined a tropical linear space to be a certain type of polyhedral complex (2008).
Tropical linear spaces have *tropical Plücker coordinates* which satisfy the tropical analogue of the Plücker relations.

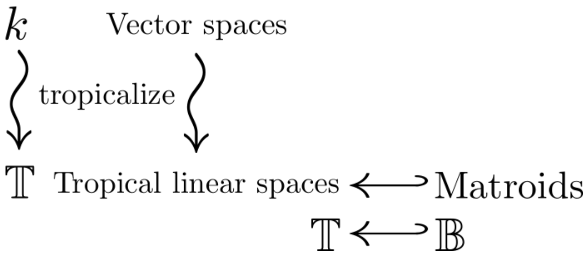
Theorem (Fink 2014)

Tropical linear spaces are degree 1 tropical varieties.

Summary

Our perspective:

- Tropical linear spaces $\leftrightarrow$ certain modules over $\mathbb{T}$.
- "Constant-coefficient" tropical linear spaces $\leftrightarrow$ matroids.
- Geometric properties of tropical linear spaces $\leftrightarrow$ combinatorial properties of matroids.



Module theory

Definition

A semiring S satisfies all the axioms of a ring, except for the existence of additive inverses.

Module theory

Definition

A semiring S satisfies all the axioms of a ring, except for the existence of additive inverses.

Example

The tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ has operations

- addition $a \oplus b = \max\{a, b\}$
- multiplication $a \odot b = a + b$.

$\mathbb{T}$ is an *idempotent* semiring: $a \oplus a = a$ for all $a \in \mathbb{T}$.

Module theory

Definition

A semiring S satisfies all the axioms of a ring, except for the existence of additive inverses.

Example

The tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ has operations

- addition $a \oplus b = \max\{a, b\}$
- multiplication $a \odot b = a + b$.

$\mathbb{T}$ is an *idempotent* semiring: $a \oplus a = a$ for all $a \in \mathbb{T}$.

Example

The Boolean semifield is the sub-semiring $\mathbb{B} = \{-\infty, 0\} \subseteq \mathbb{T}$.

Module theory

Definition

A module over a semiring S is an abelian monoid M with a bilinear, associative map $S \times M \to M$ such that, written multiplicatively, $1m = m$ for all $m \in M$.

Module theory

Definition

A module over a semiring S is an abelian monoid M with a bilinear, associative map $S \times M \to M$ such that, written multiplicatively, $1m = m$ for all $m \in M$.

Example

$\mathbb{T}^n$ has standard basis $e_i = (-\infty, \ldots, -\infty, 0_i, -\infty, \ldots, -\infty)$.
Note $e_i \in \mathbb{B}^n \subset \mathbb{T}^n$.

Module theory

Definition

A module over a semiring S is an abelian monoid M with a bilinear, associative map $S \times M \to M$ such that, written multiplicatively, $1m = m$ for all $m \in M$.

Example

$\mathbb{T}^n$ has standard basis $e_i = (-\infty, \ldots, -\infty, 0_i, -\infty, \ldots, -\infty)$.
Note $e_i \in \mathbb{B}^n \subset \mathbb{T}^n$.

Theorem

If M is a free $\mathbb{T}$-module, then M has a unique basis up to permutation and scaling.

Matroids

Definition

A matroid on ground set E with bases $\mathcal{B}$ is a set $\mathcal{B}$ of subsets of E which satisfies *strong basis exchange*: if $B, B' \in \mathcal{B}$ and $u \in B - B'$, then there exists $u' \in B' - B$ such that $B - u + u'$ and $B' - u' + u$ are in $\mathcal{B}$.

Cryptomorphisms of matroid axioms: independent sets, circuits (minimal dependence relations), flats, ...

Tropical Plücker relations

Tropical linear spaces are described by Plücker coordinates: $w_I \in S$ for $I \in \binom{E}{d}$ such that

$$\sum_{i \in A-B} w_{A-i} w_{B+i} = \sum_{i \neq j} w_{A-i} w_{B+i}$$

for all $A \in \binom{E}{d+1}$, $B \in \binom{E}{d-1}$, and $j \in A - B$.
When $w_I \in \mathbb{B}$: equivalent to *strong basis exchange* axiom for $\mathcal{B} = \{I : w_I \neq -\infty\}$.

What is a tropical linear space?

Definition

If $w \in S^{\binom{E}{d}}$ are tropical Plücker coordinates, the *tropical linear space* $L_w \subseteq S^E$ with coordinates w is

$$\bigcap_{J \in \binom{E}{d+1}} \{v : \sum_{i \in J} w_{J-i} v_i = \sum_{i \in J-j} w_{J-i} v_i \text{ for all } j \in J\}$$

What is a tropical linear space?

Definition

If $w \in S^{\binom{E}{d}}$ are tropical Plücker coordinates, the *tropical linear space* $L_w \subseteq S^E$ with coordinates w is

$$\bigcap_{J \in \binom{E}{d+1}} \{v : \sum_{i \in J} w_{J-i} v_i = \sum_{i \in J-j} w_{J-i} v_i \text{ for all } j \in J\}$$

If M is a matroid on E, let $L_M \subseteq \mathbb{B}^E$ denote the tropical linear space with Plücker coordinates the indicator vector of bases of M.

What is a tropical linear space?

Definition

If $w \in S^{\binom{E}{d}}$ are tropical Plücker coordinates, the *tropical linear space* $L_w \subseteq S^E$ with coordinates w is

$$\bigcap_{J \in \binom{E}{d+1}} \{v : \sum_{i \in J} w_{J-i} v_i = \sum_{i \in J-j} w_{J-i} v_i \text{ for all } j \in J\}$$

If M is a matroid on E, let $L_M \subseteq \mathbb{B}^E$ denote the tropical linear space with Plücker coordinates the indicator vector of bases of M.

Theorem (Murota 2001, matroid folklore)

If M is a matroid, then L_M is generated as a $\mathbb{B}$-module by the indicator vectors of cocircuits.

Minors, strong maps, and geometry

Minors of matroids

If M is a matroid on E and $T \subset E$, then two matroids on $E - T$ may be defined:

- deletion $M \setminus T$
- contraction M/T.

Minors, strong maps, and geometry

Minors of matroids

If M is a matroid on E and $T \subset E$, then two matroids on $E - T$ may be defined:

- deletion $M \setminus T$
- contraction M/T.

Theorem (Frenk 2013)

Let M be a matroid on E, and let T be a subset of that ground set. Then

- $L_{M/T} = L_M \cap \mathbb{T}^{E-T}$
- $L_{M \setminus T} = \pi_{E-T}(L_M)$

Minors, strong maps, and geometry

Definition

Let M be a matroid on E and N be a matroid on F. A *strong map* $f : M \to N$ is a pointed function $f : E \cup * \to F \cup *$ such that the preimage of a flat of N is a flat of M.

Minors, strong maps, and geometry

Definition

Let M be a matroid on E and N be a matroid on F. A *strong map* $f : M \to N$ is a pointed function $f : E \cup * \to F \cup *$ such that the preimage of a flat of N is a flat of M.

Theorem

Let $f : E \cup * \to F \cup *$ be a function. Define $f_* : \mathbb{T}^E \to \mathbb{T}^F$ by

$$f_*(e_i) = \begin{cases} e_{f(i)} & f(i) \neq * \\ 0 & f(i) = * \end{cases}.$$

Let M and N be matroids on E and F. Then $f : M \to N$ is a strong map if and only if the dual transformation $f_*^\vee$ satisfies $f_*^\vee(L_N) \subseteq L_M$.

Minors, strong maps, and geometry

Theorem (Higgs 1968)

Let M and N be matroids on E. Then $id : M \to N$ is a strong map if and only if there exists Q on $E \cup I$ such that $M = Q \setminus I$ and $N = Q/I$.

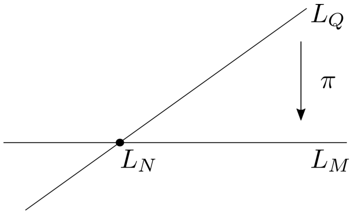
Minors, strong maps, and geometry

Theorem (Higgs 1968)

Let M and N be matroids on E. Then $\mathrm{id} : M \to N$ is a strong map if and only if there exists Q on $E \cup I$ such that $M = Q \setminus I$ and $N = Q/I$.

Theorem

Let M and N be matroids on the same ground set E. Then $L_N \subseteq L_M$ if and only if there exists L_Q such that $\pi_E(L_Q) = L_M$ and $L_Q \cap \mathbb{T}^E = L_N$.



Transversal matroids

Definition (Fink and Rincon, 2015)

A Stiefel tropical linear space is a tropical linear space whose Plücker coordinates are the maximal minors of a tropical matrix.

Stiefel tropical linear spaces over $\mathbb{B}$: transversal matroids.

Transversal matroids

For the classical Stiefel map $k^{d \times n} \dashrightarrow \bigwedge^d k^n$, the fibers are GL_d-orbits. Every fiber contains a unique matrix of the form $\begin{bmatrix} I_d & A' \end{bmatrix}$ (up to column permutation).

Theorem

A Stiefel tropical linear space L over $\mathbb{B}$ is represented by a matrix $A = \begin{bmatrix} I_d & A' \end{bmatrix}$ (up to column permutation) if and only if L corresponds to a fundamental transversal matroid.

Future directions

- Classifying general $L \subset L'$

Future directions

- Classifying general $L \subset L'$
- Matroids over hyperfields