A Module-Theoretic Approach to Matroids

Colin Crowley, Noah Giansiracusa, Joshua Mundinger

Abstract

Tropical linear spaces provide a bridge between the combinatorics of matroids, the algebra of idempotent semifields, and the geometry of tropical varieties. We expand on existing work by further developing idempotent module theory of matroids, finding new formulations of both classical matroid and tropical linear space constructions.

Idempotent algebra

A *semiring* S is just like a ring, except for that additive inverses need not exist. A semiring S is *idempotent* if $a + a = a$ for all $a \in S$. Idempotent semirings have a canonical partial order: say $a \leq b$ if $a + b = b$. A *totally ordered idempotent semifield* has a total canonical ordering. A *module* over a semiring is a monoid M with a compatible multiplication of S. Let $M^\vee = \mathrm{Hom}(M, S)$ denote the linear dual of an S-module.

Examples: $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with addition of maximum, and multiplication ordinary real addition. $\mathbb{B} = \{0, 1\}$ is the unique two-element idempotent semifield with additive identity 0 and multiplicative identity $1 \neq 0$. $\mathbb{B} \cong \{-\infty, 0\} \subseteq \mathbb{T}$.

Idempotent algebra differs from algebra over rings:

Theorem (M. '17) *Let S be an idempotent semiring with no zero divisors. Then S^n has a unique basis up to permutation and scaling.*

Bend relations

Given $v = \sum_i v_i e_i \in S^n$ and $f \in (S^n)^\vee$, f *bends* at v if $f(v) = f(\sum_{i \neq j} v_i e_i)$ for all $1 \leq j \leq n$. Given $f \in (S^n)^\vee$, the set of all $v \in S^n$ at which f bends is called the *tropical hyperplane* $V(f)$ defined by f. This replaces finding where f vanishes, which is too restrictive a condition.

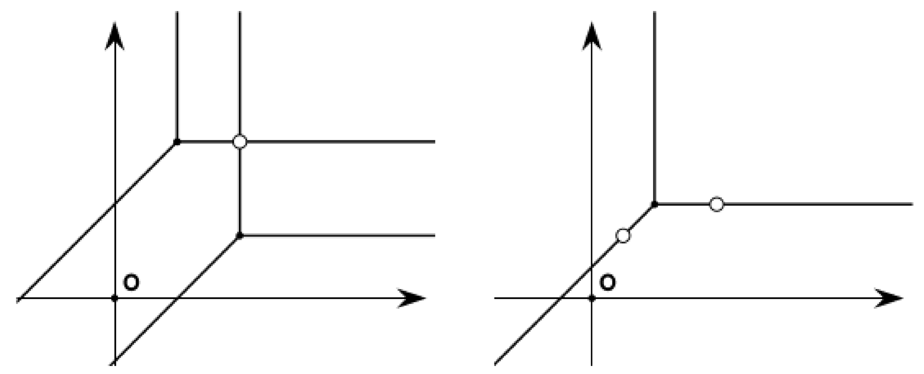


Figure: tropical lines in $\mathbb{TP}^2$. From [2], Figure 4

Matroids

A matroid is a combinatorial abstraction of notions of independence. The data for a matroid may be expressed in many different "cryptomorphic" ways. In terms of *bases*: a matroid on ground set E is a collection of bases $\mathcal{B} \subset 2^E$ satisfying the *strong basis exchange axiom*: if B_1 and B_2 are bases and $u_1 \in B_1 - B_2$, there exists $u_2 \in B_2 - B_1$ such that $B_1 - u_1 + u_2$ and $B_2 - u_2 + u_1$ are both bases.

Tropical linear space

Let S be a totally ordered idempotent semifield. A tropical Plücker vector $w \in S^{\binom{n}{d}}$ (with entries indexed by subsets of n of size d) is a vector satisfying the tropical analogue of the Plücker relations: if A and B are subsets of $[n]$ of size $d+1$ and $d-1$, then

$$\sum_{i \in A-B} w_{A-i} w_{B+i} = \sum_{i \in A-B, i \neq p} w_{A-i} w_{B+i}$$

for all $p \in A - B$. [3] If $S = \mathbb{B}$, this is equivalent to the strong basis exchange axiom for $\mathcal{B} = \{I : w_I \neq 0\}$.

Given a tropical Plücker vector w over a totally ordered idempotent semifield S, the associated tropical linear space L_w is defined as

$$L_w = \bigcap_{|J|=d+1} V(\sum_{j \in J} w_{J-j} x_j) \subseteq S^n.$$

If M is a matroid on ground set $[n]$, let L_M denote the tropical linear space in $\mathbb{B}^n$ with tropical Plücker vector the indicator vector of the bases.

Minors

Given a matroid M on ground set E and $A \subseteq E$, two related matroids on $E - A$ may be defined (combinatorially): the deletion $M \setminus A$ and contraction M/A.

Theorem ([1], Lemma 4.1.11) *If $A \subseteq E$, and M is a matroid on E, then*

$$L_{M/A} = \mathbb{B}^{E-A} \cap L_M$$
$$L_{M \setminus A} = \pi_{E-A}(L_M)$$

where π_{E-A} denotes coordinate projection onto $\mathbb{B}^{E-A}$.

Analogs of minors exist for tropical linear spaces, satisfying the above relationships.

Transversal matroids

Classically, mapping a $d \times n$ matrix to its vector of $d \times d$ minors defines a map $\mathbb{A}^{d \times n} \dashrightarrow \mathbb{P}^{\binom{n}{d}-1}$; its fibers are GL_d-orbits, and its image is all Plücker vectors.

In contrast, not every tropical Plücker vector is the vector of maximal minors of a tropical matrix; the tropical linear spaces that arise this way are Stiefel tropical linear spaces. Over $\mathbb{B}$, these are exactly transversal matroids.

Strong maps

Let M and N be matroids on ground sets E and F. A *strong map* $f : M \to N$ is a function $f : E \cup * \to F \cup *$ such that if M_* and N_* are the extensions where $*$ is a loop, then the preimage of a flat of N_* is a flat of M_*.

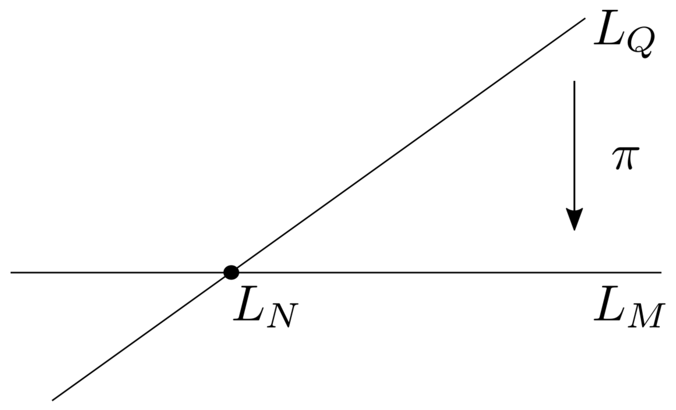
Theorem *$f : M \to N$ is a strong map if and only if for $f_* : (\mathbb{B}^E)^\vee \to (\mathbb{B}^F)^\vee$ defined by*

$$f_*(x_i) = \begin{cases} x_{f(i)} & f(i) \neq * \\ 0 & f(i) = *, \end{cases}$$

$f_*^\vee : \mathbb{B}^F \to \mathbb{B}^E$ satisfies $f_*^\vee(L_N) \subseteq L_M$.

The strong map factorization theorem gives a characterization of when the identity is a strong map, in terms of minors (cf. [4, Chapter 8]). In geometric terms, this is:

Theorem *Suppose that M and N are matroids on E such that $L_N \subseteq L_M$. Then there exists a matroid Q on $E \cup T$ such that $L_Q \cap \mathbb{B}^E = L_N$ and $\pi_E(L_Q) = L_M$.*



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