Math 267 Homework 8, Partial Solutions

Joshua Mundinger
josh@math.uchicago.edu

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8.7 Let V be of dimension 2. For $a \geq b$, show that the highest weights of the irreducible representations in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$ for $a \geq b$ are $(a+b, 0), (a+b-1, 1), \ldots, (a, b)$, each with multiplicity one.

Solution via characters: The character of the representation with highest weight $(a, 0)$ is $\phi_a = \sum_{i=0}^a z_1^i z_2^{a-i}$. Since the determinant character $z_1 z_2$ has weight $(1,1)$, the character of the representation $(a+k, k)$ is $(z_1 z_2)^k \sum_{i=0}^a z_1^i z_2^{a-i}$. The character of the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$ is thus

$$\phi_a \phi_b = (\sum_{i=0}^a z_1^i z_2^{a-i})(\sum_{j=0}^b z_1^j z_2^{b-j}).$$

The term $z_1^k z_2^\ell$ appears in this product with coefficient

$$\begin{aligned} |\{(i,j) \mid 0 \leq i \leq a, 0 \leq j \leq b, i+j = k\}| &= |\{i \mid 0 \leq i \leq a, b-k \leq i \leq k\}| \\ &= \min\{a, k\} - \max\{0, b-k\} + 1 \\ &= \min\{a, k, a+b-k, b\} + 1 \\ &= \min\{k, \ell, b\} + 1. \end{aligned}$$

This gives the decomposition

$$\phi_a \phi_b = \phi_{a+b} + z_1 z_2 \phi_{a-1} \phi_{b-1}.$$

which inductively gives the formula

$$\phi_a \phi_b = \sum_{i=0}^b \phi_{a+b-i} (z_1 z_2)^i.$$

This gives the desired irreducible decomposition of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$ by character theory.

Solution via polynomial rings: [Omitted since it was wrong.]

Geometric Solution: Consider that $\mathrm{Sym}^* V \otimes \mathrm{Sym}^* V$ is the ring of polynomials on $V^* \times V^*$. Let $\{e_1, e_2\}$ be the standard basis of V, and let $\{e^1, e^2\}$ be the dual basis of V^*. Then let

$$X = \{(a_1 e^1 + a_2 e^2, b_1 e^1) \mid b_1 \neq 0\} \subseteq V^* \times V^*$$

. Then $U \times X \to V^* \times V^*$ is injective: if $ub_1e^1 = u'b_1'e^1$, then $u = u'$ since u and u' are determined by their first row. Full injectivity follows from this calculation. Further, the image is dense. So, $(\operatorname{Sym}^* V \otimes \operatorname{Sym}^* V)^U \subseteq \mathbb{C}[X]$. We need to know which functions in the target extend to U-invariant polynomials on $V^* \times V^*$.

Now the function $(a_1e^1 + a_2e^2, b_1e^1) \mapsto b_1$ on X extends to $V^* \times V^*$ as $1 \otimes e_1 \in \operatorname{Sym}^* V \otimes \operatorname{Sym}^* V$. The function $(a_1e^1 + a_2e^2, b_1e^1) \mapsto a_1$ extends to $V^* \times V^*$ as $e_1 \otimes 1 \in \operatorname{Sym}^* V \otimes \operatorname{Sym}^* V$. The function $(a_1e^1 + a_2e^2, b_1e^1) \mapsto a_2b_1$ extends to $V^* \times V^*$ as $e_2 \otimes e_1 - e_1 \otimes e_2 \in \operatorname{Sym}^* V \otimes \operatorname{Sym}^* V$. These functions have weight $(1, 0)$, $(1, 0)$, and $(1, 1)$ respectively. The monomial functions on X are exactly of the form $a_1^i a_2^j b_1^k$, and the above computation shows that the monomial extends to $V^* \times V^*$ if and only if $j \leq k$. Thus,

$$(\operatorname{Sym}^* V \otimes \operatorname{Sym}^* V)^U = \mathbb{C}[e_1 \otimes 1, 1 \otimes e_1, e_2 \otimes e_1 - e_1 \otimes e_2].$$

These three elements indeed generate a polynomial ring by considering weights: the elements $e_1 \otimes 1$ and $1 \otimes e_1$ generate a polynomial ring with weights $(n, 0)$ for $n \geq 0$, while $e_2 \otimes e_1 - e_1 \otimes e_2$ has weight $(1, 1)$.

The weight vector

$$(e_1 \otimes 1)^i (1 \otimes e_2)^j (e_2 \otimes e_1 - e_1 \otimes e_2)^k,$$

as an element of $\operatorname{Sym}^* V \otimes \operatorname{Sym}^* V$, has degree $i + k$ in the first coordinate and $j + k$ in the second coordinate. Hence, it lies in $\operatorname{Sym}^a V \otimes \operatorname{Sym}^b V$ if and only if $i + k = a$ and $j + k = b$. It has weight $(i + j + k, k)$. Thus, the highest weight vectors in $\operatorname{Sym}^a V \otimes \operatorname{Sym}^b V$ have weights exactly $(a - b + k, k)$ where $k \in \{0, 1, \dots, b\}$.

8.12 Let (V_i, H_i) be $\mathbb{C}$-vector spaces with positive-definite Hermitian forms for $i = 1, 2$. Let $V = V_1 \oplus V_2$ and $H = H_1 - H_2$ on V, that is

$$H(v_1 \oplus v_2, w_1 \oplus w_2) = H_1(v_1, w_1) - H_2(v_2, w_2)$$

for all $v_i, w_i \in V_i$. Let $U(V, H)$ be the group of H-preserving transformations on V. Show that any compact subgroup $G \leq U(V, H)$ is contained in $g(U(V_1, H_1) \times U(V_2, H_2))g^{-1}$ for some $g \in U(V, H)$.

Solution: We first show that there are subspaces V^+, V^- of V such that $V = V^+ \oplus V^-$, V^+ and V^- are orthogonal with respect to H, $\pm H$ restricted to $V^\pm$ is positive-definite, and G preserves V^+ and V^-.

Let B be a positive-definite G-invariant Hermitian form. Then by Riesz representation, there exists $T : V \to V$ such that

$$B(Tv, w) = H(v, w)$$

for all $v, w \in V$. Since H is nondegenerate, T is an isomorphism. Since B and H are both Hermitian,

$$B(Tv, w) = H(v, w) = \overline{H(w, v)} = \overline{B(Tw, v)} = B(v, Tw),$$

so T is self-adjoint. Hence, $V = V^+ \oplus V^-$ for V^+ the positive eigenspaces of T and V^- the negative eigenspaces of T; these eigenspaces are orthogonal with respect to B. Since T takes $V^\pm$ into $V^\pm$, V^+ and V^- are orthogonal with respect to H. Since $H(v, v) = B(Tv, v)$, by decomposing V into eigenspaces for T, we see $\pm H$ is positive-definite on $V^\pm$, respectively. Since B and H are both nondegenerate and G-invariant, T is G-invariant, so V^+ and V^- are G-invariant.

Define $g_+ : V^+ \to V_1$ such that $H_1(g_+v, g_+w) = H(v, w)$ for all $v, w \in V^+$, and $g_- : V^- \to V_2$ such that $H_2(g_-v, g_-w) = -H(v, w)$ for all $v, w \in V^-$. These exist since $\pm H$ is positive-definite on $V^\pm$. Then $g = g_+ \oplus g_- : V \to V$ is an isomorphism, and $H(gv, gw) = H(v, w)$ for all $v, w \in V$ by construction. Hence $g \in U(V, H)$. Since V^+ and V^- are G-invariant, V_1 and V_2 are gGg^{-1}-invariant. Thus gGg^{-1} preserves $H|_{V_1} = H_1$ and $H|_{V_2} = H_2$, so $gGg^{-1} \subseteq U(V_1, H_1) \times U(V_2, H_2)$, as desired.