Math 267 Homework 4, Partial Solutions

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4.8 Let $G = GL_2(\mathbb{F}_p)$, $B \leq G$ the subgroup of upper triangular matrices. Let L be a degree one representation of B, and let $W = \operatorname{Ind}_B^G L$.

(1) Express $\operatorname{Res}_B W$ as a direct sum of irreducible representations of B.

(2) Compute $\langle \chi_W, \chi_W \rangle$.

Solution:

(1) By the Mackey formula (see p. 58 of Serre),

$$\operatorname{Res}_B \operatorname{Ind}_B^G L = \bigoplus_{s \in B\backslash G/B} \operatorname{Ind}_{H_s}^B L_s,$$

where L_s is the representation of $H_s = sBs^{-1} \cap B$ with underlying vector space L and the action of H_s is given by twisting the action of B on L by conjugation by s. That is, if $\rho_L : B \to GL(L)$ is the action of L, then the action of H_s on L_s is

$$\rho_{L_s}(h) = \rho_L(s^{-1}hs),$$

well-defined since $s^{-1}hs \in B$ by assumption.

We need to compute $B\backslash G/B$. A matrix $g \in G$ acts on $\mathbb{P}^1(\mathbb{F}_p)$ by sending the equivalence class $[x]$, where $x \in \mathbb{F}_p^2 \setminus 0$, to $[Ax]$. The stabilizer of $[(1,0)]$ is B, since $g(1,0) \in [(1,0)]$ if and only if the first column of g is only nonzero in its first entry. This gives a bijection $G/B \to \mathbb{P}^1$. Hence, B fixes $[(1,0)]$. The group B acts transitively on $\mathbb{P}^1 \setminus [(1,0)]$ since

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} [(\mu, 1)] = [(\lambda + \mu, 1)].$$

Hence, $B\backslash G/B$ is of size two. The double cosets are represented by B and $B\tau B$ for

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is an instance of the *Bruhat decomposition.*

Now $\tau B \tau^{-1}$ is the set of lower triangular matrices, so for $T \leq G$ the subgroup of diagonal matrices, $\tau B \tau^{-1} \cap B = T$. Hence

$$\operatorname{Res}_B \operatorname{Ind}_B^G L = L \oplus \operatorname{Ind}_T^B L_\tau.$$

Now we compute $\operatorname{Ind}_T^B L_\tau$. If we restrict to T, again by the Mackey formula

$$\operatorname{Res}_T \operatorname{Ind}_T^B L_\tau = \sum_{s \in T \backslash B / T} \operatorname{Ind}_{K_s}^T (L_\tau)_s$$

for $K_s = sTs^{-1} \cap T$. For $U \leq B$ the subgroup of matrices with 1 on the diagonal, we have $B/T = U$. For

$$u = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in U, \qquad t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in T,$$

$$tu = \begin{pmatrix} \alpha & \alpha\lambda \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & \alpha\lambda\beta^{-1} \\ 0 & 1 \end{pmatrix} t.$$

Hence $T \backslash B / T$ has two elements, T and TxT for x a Jordan block with eigenvalue 1. We may compute $xTx^{-1} \cap T = \mathbb{F}_p^\times$, the scalar matrices in G. As this subgroup is central, the twisted representation $(L_\tau)_x$ is just the restriction of L_τ to $\mathbb{F}_p^\times$. Thus,

$$\operatorname{Res}_T \operatorname{Ind}_T^B L_\tau = L_\tau \oplus \operatorname{Ind}_{\mathbb{F}_p^\times}^T \operatorname{Res}_{\mathbb{F}_p^\times} L_\tau.$$

By repeatedly applying Frobenius reciprocity, we see

$$\begin{aligned} \operatorname{Hom}_B(\operatorname{Ind}_T^B L_\tau, \operatorname{Ind}_T^B L_\tau) &\cong \operatorname{Hom}_T(\operatorname{Res}_T \operatorname{Ind}_T^B L_\tau, L_\tau) \\ &\cong \operatorname{Hom}_T(L_\tau, L_\tau) \oplus \operatorname{Hom}_T(\operatorname{Ind}_{\mathbb{F}_p^\times}^T \operatorname{Res}_{\mathbb{F}_p^\times} L_\tau, L_\tau) \\ &\cong \operatorname{Hom}_T(L_\tau, L_\tau) \oplus \operatorname{Hom}_{\mathbb{F}_p^\times}(\operatorname{Res}_{\mathbb{F}_p^\times} L_\tau, \operatorname{Res}_{\mathbb{F}_p^\times} L_\tau). \end{aligned}$$

As $\operatorname{Res}_{\mathbb{F}_p^\times} L_\tau$ is a character of $\mathbb{F}_p^\times$, the latter space is one-dimensional. We conclude that $\operatorname{Ind}_T^B L_\tau$ is the sum of two distinct irreducible representations over B. Tracing through the above identifications shows one of those irreducible representations restricts to L_τ over T, and the other is of dimension $p-1$. As L_τ is one-dimensional and $B = T[B,B]$, this uniquely determines the degree 1 summand, which we will still denote by L_τ.

(2) By Frobenius reciprocity,

$$\begin{aligned} \langle \chi_W, \chi_W \rangle &= \dim \operatorname{Hom}_G(\operatorname{Ind}_B^G L, \operatorname{Ind}_B^G L) \\ &= \dim \operatorname{Hom}_B(\operatorname{Res}_B \operatorname{Ind}_B^G L, L). \end{aligned}$$

By our description of the irreducible decomposition of $\operatorname{Res}_B \operatorname{Ind}_B^G L$ from (1), this is one if $L \ncong L_\tau$ as B-representations, and two if $L \cong L_\tau$.