Math 267 Homework 2, Partial Solutions

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2.4 Let $V = C(G, \mathbb{C})$. Precomposition by left multiplication gives a representation $G \to GL(V)$, with $(g \cdot f)(x) = f(g^{-1}x)$. Pointwise multplication by characters gives a representation $\hat{G} \to GL(V)$. Let $\mathcal{H}(G) \subseteq GL(V)$ be the subgroup generated by the images of G and $\hat{G}$. Prove V is an irreducible representation of $\mathcal{H}(G)$.

Solution: By decomposing the regular representation into irreducibles, we have for all $g \in G$ that

$$\frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(1)\chi(g) = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}.$$

Now using that G is abelian, by left-translating, we obtain that for all $g, h \in G$ that

$$\frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(h^{-1})\chi(g) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases}.$$

Now suppose that $W \subseteq V$ is a nonzero $\mathcal{H}(G)$-invariant subspace. Pick $f \in W$ and $h \in G$ such that $f(h) \neq 0$. Then

$$\left(\frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(h^{-1}) M(\chi) f \right)(g) = \begin{cases} f(h) & g = h \\ 0 & g \neq h \end{cases}.$$

Hence, the delta function δ_h at h is in W. But $\rho(g)\delta(h) = \delta_{gh}$, so W contains a basis for V.

2.5

(i) Show every member of $\mathcal{H}(G)$ can be expressed uniquely as $\lambda\rho(g)M(\chi)$ for $g \in G$, $\chi \in \hat{G}$, and λ a nth root of unity, where n is the least common multiple of the orders of elements of G.

(ii) Write down the multiplication table of $\mathcal{H}(G)$: if

$$\lambda'\rho(g')M(\chi')\lambda''\rho(g'')M(\chi'') = \lambda\rho(g)M(\chi),$$

express λ, g, χ in terms of $\lambda', g', \chi', \lambda'', g'', \chi''$.

(iii) Compute the order of $\mathcal{H}(G)$.

Solution:

(i) It suffices to show that the commutator of $\rho(g)$ for $g \in G$ and $M(\chi)$ for $\chi \in \hat{G}$ is an nth root of unity. For as the action of scalars commute with the action of both G and $\hat{G}$, this allows any product of elements of $\rho(G)$ and $M(\hat{G})$ to be reordered up to multiplication by a scalar factor. Now for $f \in C(G, \mathbb{C})$ and $x \in G$,

$$\begin{aligned}\left(\rho(g)M(\chi)\rho(g^{-1})M(\chi^{-1})f\right)(x) &= \left(M(\chi)\rho(g^{-1})M(\chi^{-1})f\right)(g^{-1}x)\\ &= \chi(g^{-1}x)(\rho(g^{-1})M(\chi^{-1})f)(g^{-1}x)\\ &= \chi(g^{-1}x)(M(\chi^{-1})f)(x)\\ &= \chi(g^{-1})f(x)\end{aligned}$$

Hence, the commutator satisfies $[\rho(g), M(\chi)] = \chi(g^{-1})$. As $g^n = 1$ and χ is a homomorphism, we conclude $\chi(g^{-1})^n = 1$, so $\chi(g^{-1})^n$ is an nth root of unity, as desired.

(ii) Say λ', λ'' are nth roots of unity, $g', g'' \in G$, and $\chi', \chi'' \in \hat{G}$. Then by part (i), $\rho(g'')M(\chi') = \chi(g'')^{-1}M(\chi')\rho(g'')$. So

$$\lambda'\rho(g')M(\chi')\lambda''\rho(g'')M(\chi'') = \lambda'\lambda''\chi(g'')\rho(g'g'')M(\chi'\chi'').$$

(iii) We will show that if λ is an nth root of unity, then $\lambda \in \mathcal{H}(G)$. This shows that every transformation of the form $\lambda\rho(g)M(\chi)$ for $\lambda^n = 1$, $g \in G$, and $\chi \in \hat{G}$ is in $\mathcal{H}(G)$. By the uniqueness in part (ii), such transformations are in bijection with $\{n\text{th roots of unity}\} \times G \times \hat{G}$, so the cardinality of $\mathcal{H}(G)$ is $n|G|^2$.

Let λ be an nth root of unity. By the structure theory for finite abelian groups, G has a cyclic direct summand of order equal to the least common multiple of orders of elements of G. Say $G = H \oplus \langle g \rangle$ where $g \in G$ is order n. A character of H and a character of $\langle g \rangle$ sum uniquely to a character of G. So there exists a character χ of G satisfying $\chi(g) = \lambda$. Then $[\rho(g), M(\chi)]^{-1} = \lambda \in \mathcal{H}(G)$.