Math 267 Homework 1, Partial Solutions

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Let S_X denote the group of permutations of a set X, and let $C(X^r, k)$ denote the set of functions from X^r to a field k.
1.2 Prove that the space of S_X-invariant vectors in $C(X^r, k)$ has dimension

1. 1 when $r = 1$ and X is nonempty
2. 2 when $r = 2$ and $|X| \geq 2$
3. 5 when $r = 3$ and $|X| \geq 3$
4. 15 when $r = 4$ and $|X| \geq 4$.

Solution: Given positive integers r and k, let $P_{r,k}$ denote the set of equivalence relations on $[r] = \{1, 2, \ldots, r\}$ with at most k equivalence classes. This is the same as the set of partitions of $[r]$ with at most k cells. Let $n = |X|$, and define $\tilde{f} : X^r \to P_{r,n}$ by sending $\mathbf{x} = (x_1, \ldots, x_r)$ to the equivalence relation $\sim_{\mathbf{x}}$ defined by $i \sim_{\mathbf{x}} j$ if and only if $x_i = x_j$. Then for $\mathbf{x} \in X^r$ and $\sigma \in S_X$,

$$\tilde{f}(\sigma\mathbf{x}) = \tilde{f}(\mathbf{x})$$

since $x_i = x_j$ if and only if $\sigma(x_i) = \sigma(x_j)$. Hence $\tilde{f}$ descends to a function

$$f : X^r/S_X \to P_{r,n}$$

defined by sending $[\mathbf{x}] \mapsto \sim_{\mathbf{x}}$.

The function f is injective: It suffices to show that if $\tilde{f}(\mathbf{x}) = \tilde{f}(\mathbf{y})$, then $\mathbf{x}$ and $\mathbf{y}$ are in the same orbit under S_X. Let $A \subseteq X$ be the elements of X appearing in $\mathbf{x}$, and $B \subseteq X$ be the elements of X appearing in $\mathbf{y}$. Sending $x_i \in A$ to the corresponding $y_i \in B$ defines a bijection $A \to B$, as the condition $\sim_{\mathbf{x}}=\sim_{\mathbf{y}}$ ensures this map and its inverse $y_i \in B \mapsto x_i \in A$ are well-defined. Since A is finite, we can extend the bijection $A \to B$ to a bijection $\sigma : X \to X$, which is an element of S_X. Then $\sigma\mathbf{x} = \mathbf{y}$, as desired.

The function f is surjective: Given a equivalence relation $\sim$, with at most n equivalence classes $A_1, \ldots, A_k$, pick distinct elements $z_1, \ldots, z_k \in X$, and define $\mathbf{x} = (x_1, \ldots, x_r) \in X^r$ by setting $x_i = z_j$ if $i \in A_j$. Since $z_1, \ldots, z_k$ are all distinct, $\tilde{f}(\mathbf{x}) = \sim$.

Now let us compute $P_{r,n}$ when $r \leq 4$ and $n \geq r$. When $n \geq r$, every partition of $[r]$ is a partition with at most n cells. If $r = 1$, then there is one partition of [1]. If $r = 2$, then there are two partitions of [2]: one with two cells and one with one cell.

If $r = 3$, then there are five partitions of [3]. Let us count them by the number of cells. There is one partition with one cell. If a partition of [3] has two cells, then one is of size two and one is of size one. There are three choices for the cell of size one, and these enumerate all such partitions. There is one partition with three cells; $\{\{1\}, \{2\}, \{3\}\}$.

If $r = 4$, then there are fifteen partitions of [4]. There is one partition with one cell. If a partition of [4] has two cells, the cells are either size 1 and 3 or size 2 and 2. In the former case, there are four possibilities; in the latter case, there are $\frac{1}{2}\binom{4}{2}$ possibilities, since we are choosing subsets of size 2 from [4], and then identifying a subset with its complement. If a partition of [4] has three cells, then the cells are size 1, 1, and 2. There are $\binom{4}{2}$ ways to choose the cell of size two, and these enumerate all such possibilities. There is one partition of [4] with four cells. This gives $1 + 4 + 3 + 6 + 1 = 15$ total partitions.

Solution 2: Let's use character theory to compute the dimension of the space of invariants when X is finite. Given a permutation $\sigma \in S_X$, let $F(\sigma)$ denote the number of fixed points of σ. The character of the representation $C(X, k)$ is F, and thus the character of $C(X^r, k) = C(X, k) \otimes C(X, k) \otimes \cdots C(X, k) = C(X, k)^{\otimes r}$ is F^k. Hence

$$\dim C(X^r, k)^{S_X} = \frac{1}{|S_X|} \sum_{\sigma \in S_X} F(\sigma)^r.$$

Let us count the number of permutations with exactly j fixed points. They are in bijections with subsets of X of size j and permutations of the complement of that set with no fixed points. Permutations with no fixed points are known as *derangements.* By an inclusion-exclusion argument, the number of derangements on m symbols is

$$m! \sum_{i=0}^{m} \frac{(-1)^i}{i!}.$$

Thus, if $n = |X|$ and $j \leq n$,

$$|\{\sigma \in S_X \mid F(\sigma) = j\}| = \binom{n}{j}(n-j)! \sum_{i=0}^{n-j} (-1)^i \frac{1}{i!} = n! \sum_{i=0}^{n-j} (-1)^i \frac{1}{i!j!}.$$

Hence,

$$\begin{aligned}
\dim C(X^r,k)^{S_X} &= \frac{1}{n!}\sum_{j=0}^{n} |\{\sigma \in S_X \mid F(\sigma)=j\}| j^r \\
&= \frac{1}{n!}\sum_{j=0}^{n}\left(n!\sum_{i=0}^{n-j}\frac{(-1)^i}{i!j!}\right) j^r \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{(-1)^i j^r}{i!j!}.
\end{aligned}$$

This gives an explicit formula for the dimension of the space of invariants, but it is not clear from this formula that the dimension of $C(X^r,k)^{S_X}$ and $C(Y^r,k)^{S_Y}$ when $|X|$ and $|Y|$ are both at least r.

1.3 Let $d(r,X) = \dim C(X^r,k)^{S_X}$. Compute $d(r,X) - d(r,Y)$ when $|X| \geq r$ and $|Y| = r - i$ for $i = 1,2,3$.

Solution: By Problem 1.2, we have $d(r,X) = |P_{r,|X|}|$. Hence if $|X| \geq r$ and $|Y| = r-i$, $d(r,X) - d(r,Y)$ is the number of partitions of $[r]$ with more than $r-i$ cells. For $i=1$, there is exactly one partition of $[r]$ with r cells, where every cell is a singleton. So if $|Y| = r-1$, $d(r,X) - d(r,Y) = 1$.

For $i=2$, there are exactly $\binom{r}{2}$ partitions of $[r]$ with exactly $r-1$ cells, since one cell must of size two and the rest singletons, and such partitions are in bijection with choices of a subset of size two of $[r]$. So if $|Y| = r-2$, $d(r,X) - d(r,Y) = 1 + \binom{r}{2}$.

For $i=3$, we must count the number of partitions with exactly $r-2$ cells. Such a partition must either have one cell of size three and $r-3$ of size one, or two of size two and $r-4$ of size one. The former are in bijection with subsets of r of size three, and the latter are in bijection with choices of a subset of size four and a partition of that set of four into sets of size two. There are $\binom{r}{3} + \binom{r}{4}\left(\frac{1}{2}\binom{4}{2}\right) = \binom{r}{3} + 3\binom{r}{4}$ of these. So if $|Y| = r-3$, $d(r,X) - d(r,Y) = 1 + \binom{r}{2} + \binom{r}{3} + 3\binom{r}{4}$.