# Representation theory of $U(n)$

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**Abstract**

These notes supplement the lectures of Math 267, Fall 2019. We present the key points of the representation theory of $U(n)$. The starting point for these notes is the Peter-Weyl theorem, in the special case of $U(n)$. These notes are infused with representation theory as I learned it from Victor Ginzburg. I make no claim to originality.

There is no royal road to geometry.

Euclid

All representations are assumed to be complex and finite-dimensional.

**Theorem 0.1** (Peter-Weyl)**.** *Let* $\operatorname{Irr} U(n)$ *denote the set of irreducible representations of* $U(n)$*. For* $W \in \operatorname{Irr} U(n)$*, let* $\theta_W : W^* \otimes W \to C(U(n), \mathbb{C})$ *be defined by the following formula: for* $w \in W, \phi \in W^*, g \in U(n)$*,*

$$\theta_W(\phi, w)(g) = \phi(\rho_W(g)w).$$

*Then with respect to the inner product of integrating against Haar measure, the images of* $\theta_W$ *for* $W \in \operatorname{Irr} U(n)$ *are orthogonal, and their image is dense in* $C(U(n), \mathbb{C})$*. Hence,*

$$\widehat{\bigoplus_{W \in \operatorname{Irr} U(n)}} W^* \otimes W = C(U(n), \mathbb{C}).$$

## 1 The Lie algebra

In this section, we expose the Lie algebra, and the relationship between representations of a group and its Lie algebra.

## 1.1 The exponential map for $GL(V)$ and differentiating a representation

Let $V$ be a vector space over $\mathbb{C}$.

**Definition 1.1.** The exponential map $\exp : \operatorname{End} V \to GL(V)$ is defined by

$$\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

The sum in the definition of the exponential map converges absolutely for any $X \in \operatorname{End} V$: pick an operator norm $\|\cdot\|$ which satisfies $\|XY\| \leq \|X\|\|Y\|$ for all $X, Y \in \operatorname{End} V$, and then use the same argument as in one variable. We also have the power series for the logarithm

$$\log(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n.$$

By the Inverse Function Theorem, there are neighborhoods $A \ni 0$ in $\operatorname{End} V$ and $B \ni 1$ in $GL(V)$ such that

$$\exp : A \cong B : \log$$

are inverses. In fact they are diffeomorphisms, or even stronger, biholomorphisms.

We wish to understand the representations of (subgroups of) $GL(V)$ by differentiating their representations. The tangent space to $1 \in GL(V)$ is $\operatorname{End} V$. The exponential map gives convenient curves along which to compute the derivative, since for $X \in \operatorname{End} V$,

$$\frac{d}{dt} \exp(tX)|_{t=0} = X.$$

To compute the derivative of $\rho : GL(V) \to GL(W)$, we need to differentiate $t \mapsto \rho(\exp(tX))$.

**Definition 1.2.** The derivative of continuous $\rho : GL(V) \to GL(W)$ is given by $d\rho(X) = \frac{d}{dt}\rho(\exp(tX))|_{t=0}$.

The following proposition tells us that this works, even if $\rho$ is only known to be continuous:

**Proposition 1.3.** *If $a : \mathbb{R} \to GL(V)$ is a continuous group homomorphism, then there is $X \in \operatorname{End} V$ such that*

$$a(t) = \exp(tX)$$

*for all $t \in \mathbb{R}$.*

*Proof.* For $t$ sufficiently small, $\log(a(t))$ is defined, say for $|t| \leq 1/n$. As power series $\log(XY) = \log(X) + \log(Y)$, and hence the identity holds also if $X$ and $Y$ are commuting matrices. Thus, $\log \circ a$ is continuous and satisfies $\log \circ a(t+s) = \log \circ a(t) + \log \circ a(s)$ for $s, t$ sufficiently small. This implies that $\log \circ a$ is linear, so that $\log \circ a(t) = tX$ for $|t| \leq 1/n$. Then for general $t$, pick $m$ such that $|t/m| \leq 1/n$. Then

$$a(t) = a(t/m)^m = \exp((t/m)X)^m = \exp(tX).$$

This proves the theorem. $\square$

Thus, for $\rho : GL(V) \to GL(W)$, $t \mapsto \rho(\exp(tX))$ is a continuous group homomorphism, and hence is of the form $\exp(td\rho(X))$ for some $d\rho(X) \in \operatorname{End} W$. We have thus proved

$$\rho(\exp(tX)) = \exp(td\rho(X)) \tag{1}$$

for all $X \in \operatorname{End} V$ and $\rho : GL(V) \to GL(W)$ continuous.

**Remark 1.4.** Since exp and log are local diffeomorphisms, this can be boosted to show that all such $\rho$ are in fact smooth!

One basic property of the derivative is that it preserves the commutator. In fact, the proof reveals the commutator on $\operatorname{End} V$ is the infinitesimal version of the commutator on $GL(V)$.

**Definition 1.5.** The bracket $[-,-]$ on $\operatorname{End} V$ is the map defined by $[X, Y] = XY - YX$.

**Lemma 1.6.** *If $\rho : GL(V) \to GL(W)$ is smooth, then*

$$d\rho([X, Y]) = [d\rho(X), d\rho(Y)].$$

*Proof.* For $t \in \mathbb{R}$, consider $e^{tX} Y e^{-tX} \in \operatorname{End} V$. I claim

$$d\rho(e^{tX} Y e^{-tX}) = \rho(e^{tX}) d\rho(Y) \rho(e^{-tX}). \tag{2}$$

For

$$\begin{aligned}\exp(sd\rho(e^{tX}Ye^{-tX})) &= \rho(\exp(se^{tX}Ye^{-tX}))\\ &= \rho(e^{tX}\exp(sY)e^{-tX})\\ &= \rho(e^{tX})e^{sd\rho(Y)}\rho(e^{-tX}).\end{aligned}$$

Now differentiate (2) with respect to $t$. Since $\rho$ is smooth, the derivative is linear, so

$$\frac{d}{dt}d\rho(e^{tX}Ye^{-tX})\bigg|_{t=0} = d\rho(XY - YX).$$

On the other hand, by (1)

$$\frac{d}{dt}\rho(e^{tX})d\rho(Y)\rho(e^{-tX})\bigg|_{t=0} = d\rho(X)d\rho(Y) - d\rho(Y)d\rho(X),$$

as desired. $\square$

Now we come to a key theorem.

**Theorem 1.7.** *Let $\rho : GL(V) \to GL(W)$ be a continuous representation, and $d\rho : \mathrm{End}(V) \to \mathrm{End}(W)$ be its derivative. Then for $W' \leq W$, the following are equivalent:*

1. *$\rho(g)W' \subseteq W'$ for all $g \in GL(V)$;*
2. *$d\rho(X)W' \subseteq W'$ for all $X \in GL(V)$.*

*Proof.* Since $W'$ is a linear space, differentiating curves in $W'$ gives derivatives in $W'$. So if $\rho(\exp(tX))W' \subseteq W'$ for all $X \in \mathrm{End}(V)$, then by using (1) and differentiating, we obtain $d\rho(X)W' \subseteq W'$. This shows 1 implies 2.

To obtain the reverse, we use the lemma:

**Lemma 1.8.** *Let $G$ be a connected topological group, and $U$ a neighborhood of the identity. Then $U$ generates $G$.*

*Proof.* By replacing $U$ with $U \cap U^{-1}$, we may assume $U$ is closed under inverses. Then $G^0 = \cup_{n\geq 1}U^n$ is the subgroup generated by $U$. I claim that $G^0$ is both open and closed. It is open since $U^n$ is open for all $n$: it is the union

$$U^n = \cup_{g\in U}gU^{n-1}.$$

I claim it is also closed. For suppose that $x$ is a limit point of $\cup_{n\geq 1}U^n$. Then $xU$ is a neighborhood of $x$, so it intersects $U^n$ for some $n$. Since $U$ is closed under inverses, this shows $x \in U^{n+1}$.

Thus $G^0$ is a connected component of $G$, and thus equals $G$. $\square$

Now we resume the proof of the Theorem. By applying the Inverse Function Theorem to exp, there is a neighborhood $B$ of $1 \in GL(V)$ in the image of the exponential map. The group $GL(V)$ is connected since it is the complement of the complex hypersurface $\{\det = 0\}$. Hence, $GL(V)$ is generated by $B$. Now if $\exp(X) \in B$ and $w' \in W$,

$$\rho(\exp(X))w' = \sum_{n=0}^{\infty} \frac{1}{n!} d\rho(X)^n w' \in W'.$$

Thus, $W'$ is stable under $B$, and thus under all of $GL(V)$. □

Theorem 1.7 tells us that if we want to check whether $W' \leq W$ is a subrepresentation of $GL(V)$, it suffices to check whether it is a subrepresentation of the Lie algebra, that is, is closed under $d\rho(\operatorname{End} V)$. In fact, a stronger statement is true: the category of representations of $GL(V)$ is a full and faithful subcategory of representations of $\operatorname{End} V$. We haven't said what a representation of $\operatorname{End} V$ is yet, but nonetheless we can make a precise statement of an equivalence in Theorem 1.11.

**Proposition 1.9.** *Let $\rho : GL(V) \to GL(W)$ be a continuous representation. Then $w \in W$ has $\rho(g)w = w$ for all $g \in GL(V)$ if and only if $d\rho(X)w = 0$ for all $X \in \operatorname{End} V$.*

*Proof.* If $\rho(g)w = w$ for all $g \in GL(V)$, then differentiating gives $d\rho(\operatorname{End} V)w = 0$. Conversely, if $d\rho(\operatorname{End} V)w = 0$, then for $X \in \operatorname{End} V$,

$$\rho(\exp X)w = \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} d\rho(X)^n\right) w = w.$$

Since $GL(V)$ is generated by exponentials, we obtain the result. □

To upgrade this, we need to know how to differentiate the action on tensor products and duals.

**Lemma 1.10.** *Let $W$ and $W'$ be representations of $GL(V)$. Then the representation on the tensor product $W \otimes W'$ is given by*

$$d\rho_{W \otimes W'} = 1 \otimes d\rho_{W'} + d\rho_W \otimes 1,$$

*that is $d\rho_{W \otimes W'}(X)(w \otimes w') = w \otimes d\rho(X)w' + d\rho(X)w \otimes w'$. The representation on the dual $W^*$ is given by*

$$d\rho_{W^*} = -d\rho_W^*,$$

*that is $d\rho_{W^*}(X)\phi = -\phi \circ d\rho(X)$.*

*Proof.* For the first calculation, use the product rule. For the second, use the chain rule and that the derivative of the inverse on $GL(V)$ is $-1$. $\square$

**Theorem 1.11.** *Let $\rho$ and $\rho'$ be representations of $GL(V)$ on $W$ and $W'$. Let $f : W \to W'$ be a linear map. Then $\rho'(g)f = f\rho(g)$ for all $g \in GL(V)$ if and only if $d\rho'(X)f = fd\rho(X)$ for all $x \in \operatorname{End} V$.*

*Proof.* Consider $\operatorname{Hom}(W, W') = W' \otimes W$. Then the action of $GL(V)$ on $\operatorname{Hom}(W, W')$ is given by $g : T \mapsto \rho'(g)T\rho(g^{-1})$. By Lemma 1.10, the derivative of this representation $d\rho$ is given by

$$X : T \mapsto d\rho'(X)T - Td\rho(X).$$

Then applying Proposition 1.9 gives $T\rho(g) = \rho'(g)T$ for all $g \in GL(V)$ if and only if $Td\rho(X) = d\rho'(X)T$ for all $X \in \operatorname{End} V$. $\square$

This implies Proposition 1.9 by looking at maps in from the trivial representation. Theorem 1.11 tells us that to understand maps between representations, it is good enough to understand maps that respect the derivatives of those representations.

## 1.2 The unitary group and the general linear group

This theorem was proved in class by Madhav.

**Theorem 1.12.** *Let $W$ be a representation of $U(n)$. Then for $V = \mathbb{C}^n$, $W$ extends to a representation of $GL(V)$; indeed, $W \subseteq \oplus_{i=1}^{k} V^{\otimes r} \otimes (V^*)^{\otimes r'}$.*

*Proof.* The matrix entries of $U(n)$ on the tensor product $V^{\otimes r} \otimes (V^*)^{\otimes r'}$ are exactly polynomials on $U(n)$ in the matrix entries $z_{ij}$ and their complex conjugates $\overline{z}_{ij}$, bihomogeneous of degree $r$ in $z_{ij}$ and $r'$ in $\overline{z}_{ij}$. Hence, the span of the matrix entries of irreducible representations of $U(n)$ contained in such tensor products is $\mathbb{C}[z_{ij}, \overline{z}_{ij}] \subseteq C(U(n), \mathbb{C})$. Polynomials are dense in continuous functions with the supremum norm by the Stone-Weierstrass theorem. So, $U(n)$ has no other irreducible representations. The full result then follows from complete reducibility. $\square$

Now such a representation is not an arbitrary representation of $GL(V)$, for they respect the complex linear structure. This is captured by the following definition:

**Definition 1.13.** A representation $\rho : GL(V) \to GL(W)$ is *holomorphic* if it is smooth and $d\rho(zX) = zd\rho(X)$ for all $z \in \mathbb{C}$.

Then $V$ is a holomorphic representation of $GL(V)$, and by Lemma 1.10, all tensor products of $V$ and its dual are holomorphic.

**Corollary 1.14.** *Every representation of $U(n)$ extends to a holomorphic representation of $GL(V)$.*

Now an holomorphic function on $\mathbb{C}$ in the usual sense is determined by its Taylor series, which may be computed from its restriction to $\mathbb{R}$. This idea may be powerfully applied to the representation theory of $U(n)$.

**Definition 1.15.** The Lie algebra $\mathfrak{u}(n) \subseteq \operatorname{End} \mathbb{C}^n$ is the space of skew-adjoint transformations on $V$ with respect to the Hermitian product. That is, $\mathfrak{u}(n) = \{X \in \operatorname{End} \mathbb{C}^n \mid X^* = -X, \}$, where $X^*$ denotes the Hermitian adjoint.

**Theorem 1.16.** *Let $W$ be a holomorphic representation of $GL(V)$. Then for $W' \leq W$, the following are equivalent:*

1. *$W'$ is stable under $GL(V)$;*
2. *$W'$ is stable under $U(n)$;*
3. *$W'$ is stable under $\mathfrak{u}(n)$;*
4. *$W'$ is stable under $\operatorname{End} V$.*

*Proof.* Theorem 1.7 tells us that 4 implies 1. 1 implies 2 since $U(n) \subseteq GL(V)$. 2 implies 3 by differentiating.

Now $i\mathfrak{u}(n)$ is the space of self-adjoint transformations on $V$, as $X^* = -X$ if and only if $(iX)^* = (-i)X^* = iX$. Then every transformation in $\operatorname{End} V$ may be written uniquely as the sum of a self-adjoint and skew-adjoint transformation. so $\operatorname{End} V = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ as real vector spaces. As $W$ is holomorphic, $d\rho(\mathfrak{u}(n))W' \subseteq W'$ also implies $d\rho(i\mathfrak{u}(n))W' \subseteq W'$, so that $d\rho(\operatorname{End} V)W' \subseteq W'$, showing that 3 implies 4. $\square$

Analogues of Proposition 1.9 and Theorem 1.11 also hold relating holomorphic representations of $GL(V)$ to their restriction to $U(n)$. It is in this sense that studying representations of $U(n)$ is the same as studying (holomorphic) representations of $GL(V)$. This equivalence also tells us that the extension of a representation of $U(n)$ to a holomorphic representation of $GL(V)$ is unique.

# 2 Highest weight theory

## 2.1 Torus, Borel, unipotent radical

Now, we fix a basis $e_1, \ldots, e_n$ for $V$ over $\mathbb{C}$. This gives a basis $\{E_{ij}\}_{i,j=1}^n$ for $\operatorname{End} V$, where $E_{ij}e_j = e_i$.

**Definition 2.1.** The *torus* $T \subseteq GL(V)$ is the subgroup of diagonal matrices with respect to a given basis The subgroup of diagonal matrices in $U(n)$ (with respect to an orthogonal basis) is also called the torus, and is also denoted $T$. The corresponding Lie algebra is the subalgebra of diagonal matrices $\mathfrak{h} = \operatorname{span}\{E_{ii}\}_{i=1}^n \subseteq \operatorname{End} V$.

The torus of $U(n)$ is compact abelian with character group isomorphic to $\mathbb{Z}^n$. The isomorphism is given by: for $\mathbf{m} \in \mathbb{Z}^n$,

$$\chi_{\mathbf{m}}\left(\begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix}\right) = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$$

These have derivative

$$d\chi_{\mathbf{m}}\left(\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}\right) = m_1 h_1 + m_2 h_2 + \cdots + m_n h_n. \tag{3}$$

Hence, given a representation $W$ of $U(n)$, we decompose the restriction of $W$ to the torus into irreps

$$W = \bigoplus_{\mathbf{m} \in \mathbb{Z}^n} W(\mathbf{m}).$$

Now $W$ extends to a holomorphic representation of $GL(V)$, and thus we have a $\mathbb{C}$-linear action of $\operatorname{End} V$. This allows us to use operators not in $\mathfrak{u}(n)$ to analyze $W$, a powerful tool. This is fruitfully breaking symmetry.

The choice of a torus fixes a basis $\{e_1, \ldots, e_n\}$, up to reordering. We can get further by fixing an ordering, and looking at the flag corresponding to that ordering

**Definition 2.2.** The Borel $B \subseteq GL(V)$ is the subgroup of upper-triangular matrices. The unipotent radical $U \subseteq GL(V)$ is the subgroup of $B$ with 1's on the diagonal. The corresponding Lie algebras $\mathfrak{b}$ and $\mathfrak{n}$ are the upper-triangular matrices and strictly upper-trianglar matrices. Their opposites $B_-, U_-, \mathfrak{b}_-, \mathfrak{n}_-$ are the lower-triangular analogues.

**Remark 2.3.** These are not the optimal definitions if one wishes to study other groups.

**Proposition 2.4.** *Let $G$ be one of the torus, Borel, unipotent radical, or their opposites. Let $\mathfrak{g}$ be the associated Lie algebra. Then for representations $W, W'$ of $G$,*

$$\mathrm{Hom}_G(W, W') = \mathrm{Hom}_{\mathfrak{g}}(W, W').$$

*Proof.* The torus, Borel, and unipotent radical are all connected. It can be checked in each case that $\exp(\mathfrak{g}) \subseteq G$, and that $\dim \mathfrak{g} = \dim G$. Hence, the Inverse Function Theorem gives a neighborhood of $1 \in G$ in the image of $\exp(\mathfrak{g})$. Then all the proofs leading to Theorem 1.11 go through verbatim. □

In particular, vectors fixed by such a $G$ are the same as vectors annihilated by the associated $\mathfrak{g}$.

The interest in $\mathfrak{n}$ is that it raises weights in a representation of $GL(V)$, where the ordering is the lexicographic ordering. This comes from the computation that if $i \neq j$,

$$[E_{kk}, E_{ij}] = \begin{cases} E_{ij} & k = i \\ -E_{ij} & k = j \\ 0 & \text{else} \end{cases}.$$

**Proposition 2.5.** *For $W$ any holomorphic representation of $GL(V)$,*

$$d\rho(E_{ij})W(\mathbf{m}) \subseteq W(\mathbf{m} + e_i - e_j).$$

*Proof.* By (3) $w \in W(\mathbf{m})$ if and only if for all $h = \sum_{i=1}^n c_i E_{ii}$, $d\rho(h)w = (\sum_{i=1}^n c_i m_i)w$. Then for $w \in W(\mathbf{m})$,

$$\begin{aligned} d\rho(h)d\rho(E_{ij})w &= d\rho(E_{ij})d\rho(h)w + [d\rho(h), d\rho(E_{ij})]w \\ &= \sum_{i=1}^n c_i m_i d\rho(E_{ij})w + (c_i - c_j)d\rho(E_{ij})w. \end{aligned}$$

This shows $w \in W(\mathbf{m} + e_i - e_j)$. □

## 2.2 Highest weight vectors

**Definition 2.6.** For $W$ a holomorphic $GL(V)$-representation, $w \in W$ is a *highest weight vector* if $U$ fixes $w$ and $T$ acts on $w$ by a character.

By Proposition (2.4), $w$ is a highest weight vector if and only if $d\rho(\mathfrak{n})w = 0$ and $w$ is an $\mathfrak{h}$-weight vector.

**Example 2.7.** If $W = \bigwedge^k V$, then $W$ has the unique highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$. For $\bigwedge^k V$ has a basis of weight vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $i_1 < i_2 < \cdots < i_k$, which has weight $e_{i_1} + \cdots + e_{i_k} \in \mathbb{Z}^n$. These have distinct weights, so these are all of the weight vectors. Hence, a highest weight must be one of them. If $\{i_1 < i_2 < \cdots < i_k\} \neq \{1, 2, \ldots, k\}$, then pick $i = i_\ell$ such that $i-1$ is not in $\{i_1 < i_2 < \cdots < i_k\}$. Then using Lemma 1.10, if $\rho$ denotes our representation,

$$d\rho(E_{i-1,i})e_{i_1} \wedge \cdots \wedge e_{i_k} = e_{i_1} \wedge \cdots \wedge e_{i_{\ell-1}} \wedge e_{i-1} \wedge e_{i_{\ell+1}} \wedge \cdots \wedge e_{i_k} \neq 0.$$

As $E_{i-1,i} \in \mathfrak{n}$, we conlclude that $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ is not a highest weight vector. Conversely, we may compute that if $i < j$, then

$$d\rho(E_{ij})e_1 \wedge e_2 \wedge \cdots \wedge e_k = 0$$

since $d\rho(E_{ij})e_i \wedge e_j = e_i \wedge e_i = 0$, and all other terms are annihilated by $d\rho(E_{ij})$. Since $\{E_{ij}\}_{i<j}$ is a basis for $\mathfrak{n}$, we conclude that

$$d\rho(\mathfrak{n})(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = 0.$$

Thus, $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a highest weight vector. Its weight is $(1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^n$, where 1 appears $k$ times.

**Example 2.8.** If $w \in W$ and $w' \in W'$ are highest-weight vectors, then $w \otimes w' \in W \otimes W'$ is a highest weight vector. However, $W \otimes W'$ in general has other highest weight vectors. For instance, $V \otimes V = \operatorname{Sym}^2 V \oplus \bigwedge^2 V$, which has highest weight vectors $e_1^2 \in \operatorname{Sym}^2 V$ and $e_1 \wedge e_2 \in \bigwedge^2 V$.

**Theorem 2.9.** *If $W$ is an irreducible holomorphic $GL(V)$-representation, then $W$ has a unique highest-weight vector.*

*Proof.* The existence follows from that the symmetric group $S_n \subseteq GL(V)$ permutes the weight spaces $W(\mathbf{m})$, and thus there exists $\mathbf{m} \in \mathbb{Z}^d$ such that $W(\mathbf{m}) \neq 0$ and $\mathbf{m}$ is lexicographically larger than any other weight of $W$. Then by Proposition 2.5, if $i < j$, then $E_{ij}W(\mathbf{m}) = 0$ by our construction of $\mathbf{m}$. Thus $d\rho(\mathfrak{n})W(\mathbf{m}) = 0$, so all elements of $W(\mathbf{m})$ are highest weight vectors.

Now we show uniqueness. We give two proofs.

1. Observe that $\operatorname{End} V = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. I claim that if $w \in W$ is a highest weight vector, then

$$W = \operatorname{span}\{d\rho(n_1)d\rho(n_2)\cdots d\rho(n_k)w \mid n_i \in \mathfrak{n}^-\};$$

that is, $W$ is generated by $w$ under just the action of $\mathfrak{n}^-$. For let $W'$ be this span. Certainly if $x \in \mathfrak{n}^-$, then $d\rho(x)W' \subseteq W'$. Now say $x \in \mathfrak{h} \oplus \mathfrak{n}$. Then

$$\begin{aligned} d\rho(x)d\rho(n_1)\cdots d\rho(n_k)w = {} & d\rho(n_1)\cdots d\rho(n_k)d\rho(x)w \\ & + [d\rho(x), d\rho(n_1)\cdots d\rho(n_k)]w. \end{aligned}$$

We show by induction on $k$ that the above is in $W'$. Then $d\rho(x)w$ is in the span of $w$ since $w$ is highest weight. The commutator in the above equation satisfies

$$\begin{aligned} &[d\rho(x), d\rho(n_1)\cdots d\rho(n_k)] \\ &\quad = \sum_{i=1}^{k} d\rho(n_1)\cdots d\rho(n_{i-1})[d\rho(x), d\rho(n_i)]d\rho(n_{i+1})\cdots d\rho(n_k). \end{aligned}$$

Then the terms $[d\rho(x), d\rho(n_i)]d\rho(n_{i+1})\cdots d\rho(n_k)$ have less than $k$ factors of the form $d\rho(n)$, $n \in \mathfrak{n}^-$. Hence by induction these terms take $w$ into $W'$. This completes showing $W' = W$.

Now if $w$ is a highest weight vector of weight $\mathbf{m}$, then

$$W = \mathbb{C}w \oplus \bigoplus_{\mathbf{l}<\mathbf{m}} W(\mathbf{l}),$$

since the operators $d\rho(\mathfrak{n}^-)$ lower the weights. So if $w'$ is another highest weight vector of weight $\mathbf{m}'$, then $\mathbf{m} > \mathbf{m}'$. But by the same logic, $\mathbf{m}' > \mathbf{m}$, a contradiction.

2. In the homework, it was shown that $U_-TU \subseteq GL(V)$ is dense. Thus, the span of $U_-TUw$ equals the span of $GL(V)w$ for all $w \in W$, which is $W$ since $W$ is irreducible. Now if $w^* \in W$ is a highest weight vector of weight $\mathbf{m}^*$, $W = \operatorname{span} U_-TUw^* = \operatorname{span} U_-w^*$. By Proposition 2.4, a $U_-$-stable subspace is the same as an $\mathfrak{n}^-$-stable subspace, so

$$\operatorname{span} U_-w^* = \operatorname{span}\{d\rho(n_1)\cdots d\rho(n_k)w^* \mid n_i \in \mathfrak{n}^-\}.$$

Now proceed as in 1.

$\square$

Now we know that every irreducible representation has a highest weight. If $\varphi : W \to W'$ is a map of representations, then it respects the action of $T$ and $U$, and thus takes highest weight vectors to highest weight vectors (of the same weight). We have not yet proved that different irreducible representations have different highest weights. Madhav's approach to this in class was via the Borel-Weil theorem, which explicitly calculates the highest-weight vectors in $C(U(n), \mathbb{C})$. Indeed, it calculates the fixed vectors under $U_- \times U$, which is a sort of analog for $U \subseteq GL(V)$ for the group $GL(V) \times GL(V)$. Another approach is via Verma modules, which requries more algebraic overhead to use. In any case, we will not prove it here, and leave it as the following proposition:

**Proposition 2.10.** *For* $\mathbf{m} \in \mathbb{Z}^n$ *such that* $m_1 \geq m_2 \geq \cdots \geq m_n$*, there exists a unique irreducible holomorphic* $GL(V)$*-representation* $W_{\mathbf{m}}$ *with highest weight* $\mathbf{m}$*.*